

Evgeny Epelbaum, RUB

SFB 1245 Topical Lecture Week, IKP, TU Darmstadt
18.-20. August 2025

Effective theory of nuclear interactions

Concepts

Effective field theory, chiral perturbation theory, renormalization, predictive power, KSW vs Weinberg, power counting...

Methods

Effective Lagrangian, heavy-baryon expansion, perturbative calculation of the amplitude, methods to derive nuclear forces (and currents), ...



Day 1

Part I: General introduction to EFT

EFT philosophy, renormalization, power counting, construction principles...

Part II: Chiral perturbation theory

Chiral symmetry, effective Lagrangian, chiral expansion, loops, inclusion of nucleons, ...

Day 2

Part III: Pionless EFT

Resummation of the amplitude, fine tuning, renormalization conditions, RG analysis,...

Part IV: Inclusion of pions

How to include pions non-perturbatively? Long-range physics and low-energy theorems...

Day 3

Part V: From $\mathcal{L}_{\text{eff}}$ to nuclear forces

How to derive nuclear forces from the effective Lagrangian?

Part VI: Gradient flow method

How to introduce regularization in the way consistent with the symmetries?

III: Pionless EFT for 2 nucleons

Use the simplest (analytically solvable) EFT for NN to clarify the meaning of renormalization, power counting and all that...

Outline

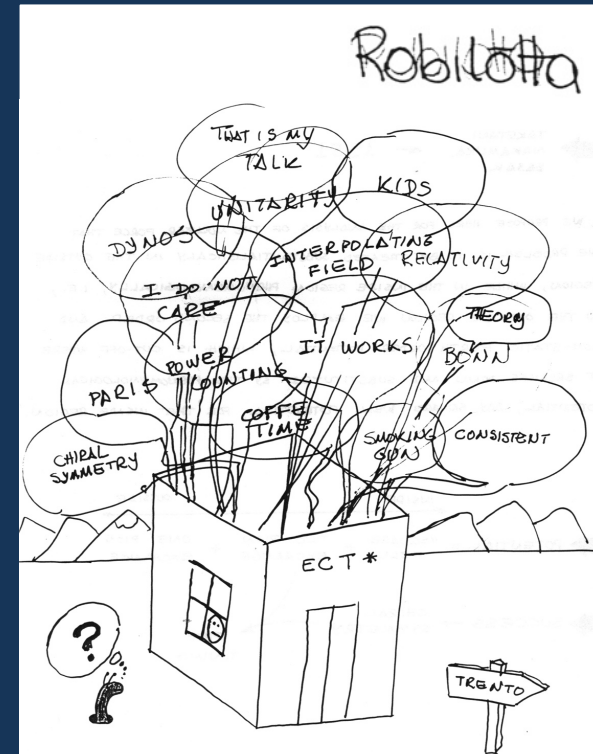
- Lippmann-Schwinger equation from Feynman diagrams
- First naive attempt and the need for fine tuning
- KSW & W power counting schemas
- Implicit renormalization

Further reading

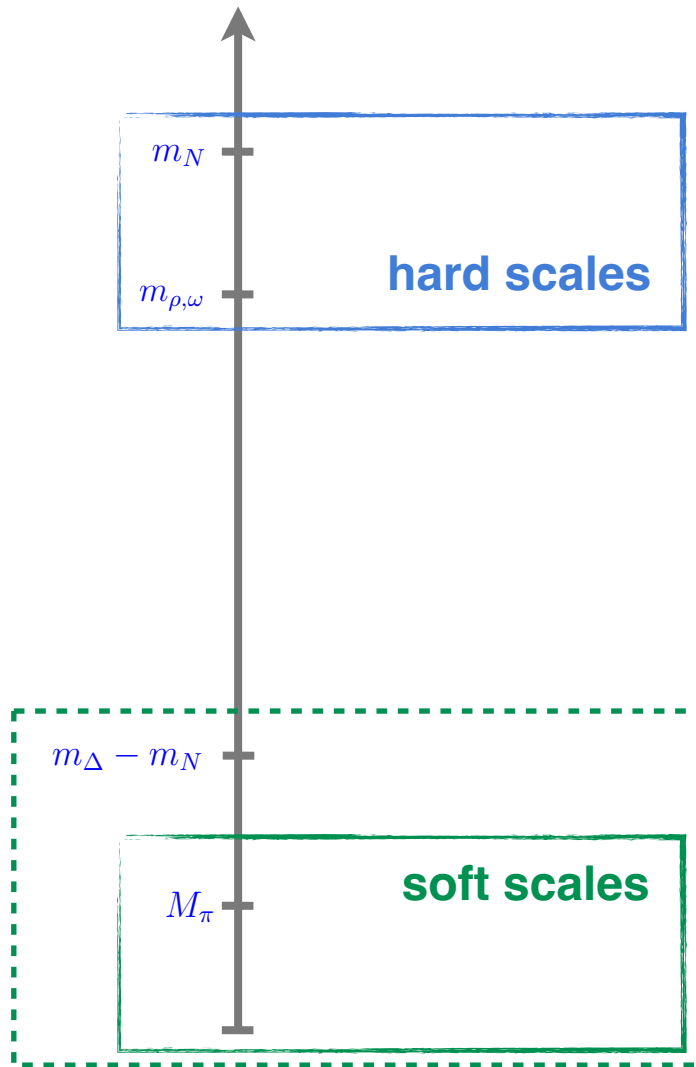
Kaplan, Savage, Wise, NPB 478 (1996) 629

EE, Gegelia, Meißner, NPB 925 (2017) 161

EE, Gegelia, Huesmann, Meißner, FBS 62 (2021) 51



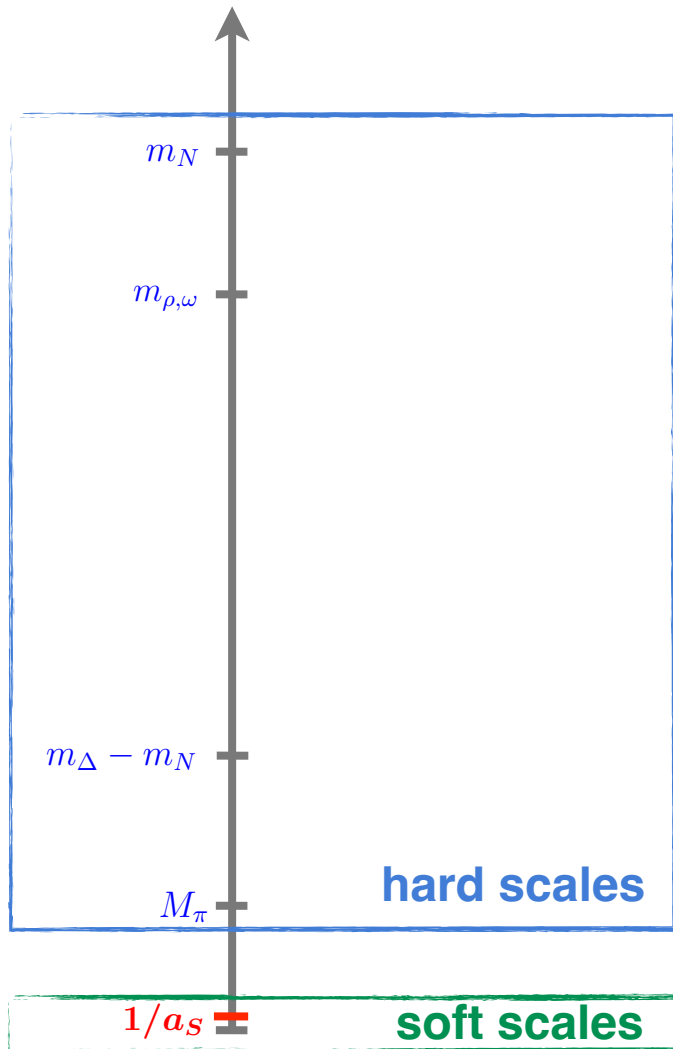
Relevant scales



- **Chiral perturbation theory** (0,1 nucleons): perturbative expansion of the amplitude in powers of

$$Q \in \left\{ \frac{M_\pi}{\Lambda}, \frac{|\vec{p}|}{\Lambda} \right\}, \quad \Lambda \sim m_\rho \sim 4\pi F_\pi \sim 1 \text{ GeV}$$

Relevant scales



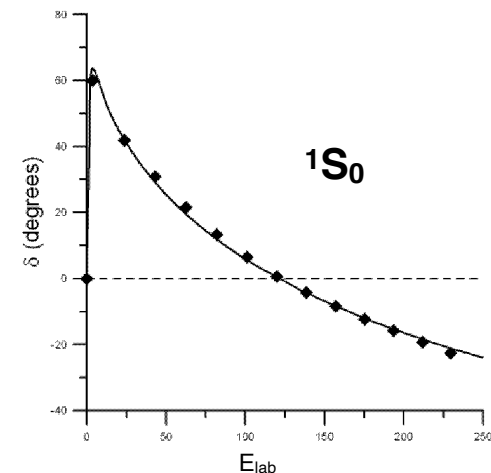
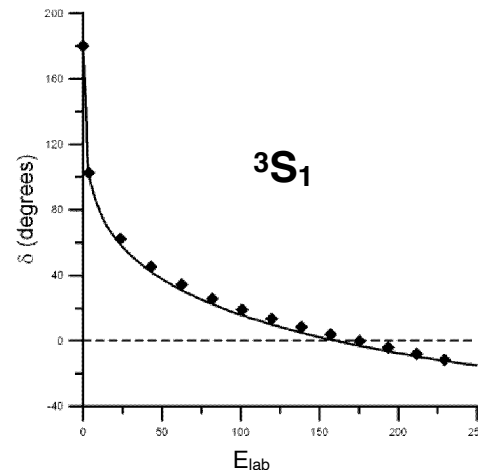
- **Chiral perturbation theory** (0,1 nucleons): perturbative expansion of the amplitude in powers of

$$Q \in \left\{ \frac{M_{\pi}}{\Lambda}, \frac{|\vec{p}|}{\Lambda} \right\}, \quad \Lambda \sim m_{\rho} \sim 4\pi F_{\pi} \sim 1 \text{ GeV}$$

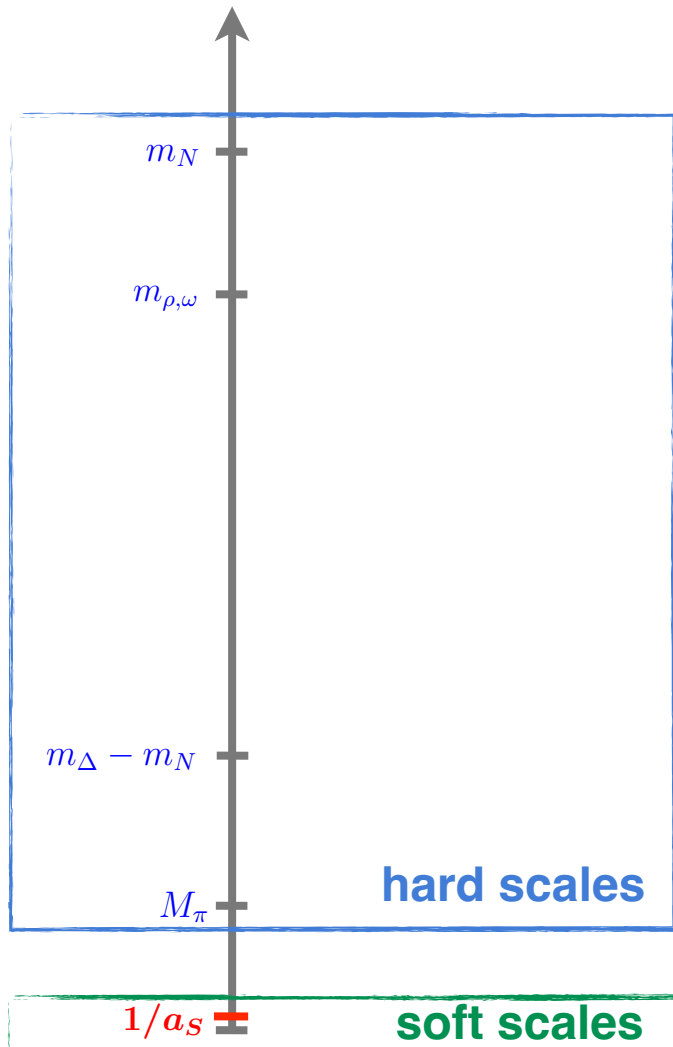
- >1 nucleons: a new very soft scale

$$1/a_S \simeq 8.5 \text{ MeV} (36 \text{ MeV}) \text{ in } {}^1S_0 ({}^3S_1)$$

has to be generated dynamically $\rightarrow$ need nonperturbative resummations: **chiral EFT**



Relevant scales



- **Chiral perturbation theory** (0,1 nucleons): perturbative expansion of the amplitude in powers of

$$Q \in \left\{ \frac{M_{\pi}}{\Lambda}, \frac{|\vec{p}|}{\Lambda} \right\}, \quad \Lambda \sim m_{\rho} \sim 4\pi F_{\pi} \sim 1 \text{ GeV}$$

- >1 nucleons: a new very soft scale

$$1/a_S \simeq 8.5 \text{ MeV} (36 \text{ MeV}) \text{ in } {}^1S_0 ({}^3S_1)$$

has to be generated dynamically $\rightarrow$ need nonperturbative resummations: **chiral EFT**

- **Pionless EFT:**

expansion of the amplitude in powers of $|\vec{p}| \sim 1/a_S$ over $\Lambda \sim M_{\pi}$

Both ERE & π -EFT yield an expansion of the amplitude in $|\vec{p}|/M_{\pi}$, have the same validity range and incorporate the same physics $\rightarrow$ **ERE $\sim \pi$ -EFT**

Pionless EFT

Pionless EFT

Kaplan, Savage, Wise, Nucl. Phys. B478 (1996) 629

The goal: design an EFT to match ERE (no predictive power for NN beyond ERE)

DoF: nonrelativistic nucleons (use the HB formalism)

Symmetries: rotational invariance, isospin symmetry, usual discrete symmetries...

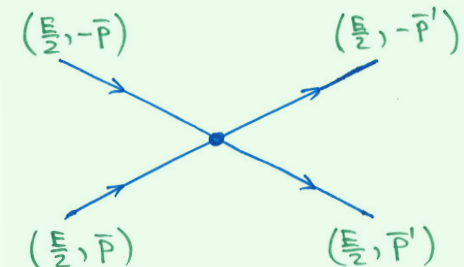
Begin with writing down the most general Lagrangian:

$$\mathcal{L} = N^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} \right) N - \frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_T (N^\dagger \vec{\sigma} N)^2 + \underbrace{\dots}_{\text{terms with } \geq 2 \text{ derivatives}}$$

Notice: $(N^\dagger \vec{\tau} N)^2$, $(N^\dagger \vec{\tau} \vec{\sigma} N)^2$ are redundant (Pauli principle). Indeed, in there are only 2 independent s-waves (1S_0 and 3S_1) in the isospin limit...

Feynman rule (ignore spin for the moment...):

$$i\mathcal{A}^{\text{tree}} = -i \underbrace{\left[C_0 + C_2 (\vec{p}^2 + \vec{p}'^2) + \dots \right]}_{\text{linear combination of } C_S, C_T}$$



Pionless EFT

$$i\mathcal{A} = \text{[Contact Diagram]} + \text{[One-loop Diagram]} + \dots$$

$$\begin{aligned}
 i\mathcal{A}^{1\text{-loop}} &= \int \frac{d^4l}{(2\pi)^4} (-i) [C_0 + C_2(\vec{p}^2 + \vec{l}^2) + \dots] \frac{i}{\frac{E}{2} + l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon} \frac{i}{\frac{E}{2} - l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon} \\
 &\quad \times (-i) [C_0 + C_2(\vec{l}^2 + \vec{p}'^2) + \dots] \\
 &= (-i) \int \frac{d^3l}{(2\pi)^3} [C_0 + C_2(\vec{p}^2 + \vec{l}^2) + \dots] \frac{1}{E - \frac{\vec{l}^2}{m_N} + i\epsilon} [C_0 + C_2(\vec{l}^2 + \vec{p}'^2) + \dots]
 \end{aligned}$$

Since loop integrals factorize, the results are trivially generalizable to any number of loops. One finds for $E = p^2/m_N$:

$$\mathcal{A}(\vec{p}', \vec{p}) = V(\vec{p}', \vec{p}) - m_N \int \frac{d^3l}{(2\pi)^3} \frac{V(\vec{p}', \vec{l}) \mathcal{A}(\vec{l}, \vec{p})}{\vec{p}^2 - \vec{l}^2 + i\epsilon}$$

sign convention for $V, \mathcal{A}$

with the potential $V(\vec{p}', \vec{p}) = -(C_0 + C_2(\vec{p}^2 + \vec{p}'^2) + \dots)$. As expected, the nonrelativistic treatment recovers the quantum mechanical Lippmann-Schwinger equation.

Pionless EFT

In the following, we focus on S-wave scattering. Utilizing the KSW notation,

$$i\mathcal{A}^{\text{tree}} = iV(p', p) = -i(C_0 + \underbrace{C_2(p^2 + p'^2)}_{\substack{\text{terms} \sim pp' \\ \text{contribute to } p\text{-waves}}} + \underbrace{\dots}_{\geq 4 \text{ derivatives}})$$

The LS equation for the half-shell amplitude in the s-waves:

$$\mathcal{A}(p', p) = V(p', p) - m \int \frac{d^3l}{(2\pi)^3} \frac{V(p', l) \mathcal{A}(l, p)}{p^2 - l^2 + i\epsilon}, \quad \text{and} \quad S_0 = 1 + i \frac{mp}{2\pi} \mathcal{A}$$

Loop integrals are UV divergent $\Rightarrow$ need regularization and renormalization...

$$m \int \frac{d^3l}{(2\pi)^3} \frac{l^{2n}}{p^2 - l^2 + i\epsilon} = \underbrace{-m \int \frac{d^3l}{(2\pi)^3} l^{2n-2}}_{=: I_{2n+1}} - \dots - \underbrace{mp^{2n-2} \int \frac{d^3l}{(2\pi)^3}}_{=: p^{2n-2} I_3} + \underbrace{mp^{2n} \int \frac{d^3l}{(2\pi)^3} \frac{1}{p^2 - l^2 + i\epsilon}}_{=: p^{2n} I(p)}$$

Cutoff regularization (DimReg + PDS correspond to $\mu \rightarrow \mu\pi/2$, $\mu_i = 0$):

$$I_n \rightarrow I_n^\Lambda = \frac{-m}{2\pi^2} \int_0^\Lambda dl l^{n-1} = \frac{-m}{2\pi^2} \int_{\mu_n}^\Lambda dl l^{n-1} + \frac{-m}{2\pi^2} \int_0^{\mu_n} dl l^{n-1} \equiv \Delta_n(\mu_n) + I_n^R(\mu_n)$$

$$I(p) \rightarrow I^\Lambda(p) = \frac{m}{2\pi^2} \int_0^\Lambda dl \frac{l^2}{p^2 - l^2 + i\epsilon} = \Delta_1(\mu) + I_1^R(\mu) - \frac{i m p}{4\pi} - \frac{m p}{4\pi^2} \ln \frac{\Lambda - p}{\Lambda + p}$$

Pionless EFT

We still need to specify renormalization conditions (= choice of subtraction scales).

Conventional wisdom suggests: $\mu, \mu_i \sim$ soft scale $\sim p \ll M_\pi$ [i.e., all loop momenta are of the order of the soft scale after renormalization...]

The amplitude to NNLO assuming NDA scaling of LECs:

$$\mathcal{A} = \underbrace{\text{order } p^0}_{-C_0} - \underbrace{\text{order } p^1}_{\hbar (-C_0)^2 I(p)} + \underbrace{\text{order } p^2}_{\hbar^2 (-C_0)^3 (I(p))^2} + \underbrace{\text{order } p^2}_{(-C_2)2p^2} + \dots$$

$$\text{Recall: } I(p) = \underbrace{\lim_{\Lambda \rightarrow \infty} \left(\frac{m}{2\pi^2} (\mu - \Lambda) \right)}_{= \Delta(\mu)} - \underbrace{\frac{m}{4\pi} \left(ip + \frac{2}{\pi} \mu \right)}_{= I^R(\mu, p)}$$

$$\text{Renormalization: } C_0 = C_0^R(\mu) + \underbrace{\hbar \delta C_{0,1}}_{-(C_0^R)^2 \Delta} + \underbrace{\hbar^2 \delta C_{0,2}}_{(C_0^R)^3 \Delta^2} + \mathcal{O}(\hbar^3)$$

Thus, finally: $\mathcal{A} = -C_0^R(\mu) - (C_0^R(\mu))^2 I^R(\mu, p) - (C_0^R(\mu))^3 (I^R(\mu, p))^2 - 2C_2 p^2 + \dots$

[If all c.t. are included, renormalization amounts to replacing $C_i \rightarrow C_i^R(\mu)$, $I_{[n]} \rightarrow I_{[n]}^R(\mu)$.]

Pionless EFT

To determine LECs, we have to match the amplitude to the ERE:

$$\mathcal{A} = \frac{4\pi}{m} \frac{1}{p \cot \delta - ip} = \frac{4\pi}{m} \frac{1}{\left[-\frac{1}{a} + \frac{1}{2}rp^2 + v_2p^4 + \dots\right] - ip}$$

Such matching is possible provided $a, r, v_i \sim \mathcal{O}(1)$. One then has:

$$\mathcal{A} = \frac{4\pi}{m} \left(-a + ia^2p + a^3p^2 - \frac{a^2r}{2}p^2 + \dots \right) \stackrel{!}{=} -C_0^R - C_0^{R^2} \underbrace{I^R(\mu, p)}_{-m/(4\pi)(2\mu/\pi + ip)} - C_0^{R^3} \left(I^R(\mu, p) \right)^2 - 2C_2p^2 + \dots$$

$$\rightarrow \begin{cases} C_0^R = \frac{4\pi a}{m} [1 + \mathcal{O}(a\mu)] \\ C_2 = \frac{\pi a^2}{m} r \end{cases} \quad \begin{array}{l} \text{[Choosing } \mu = 0, \text{ one reproduces} \\ \text{exactly the first four terms in the} \\ \text{expansion of } \mathcal{A} \dots] \end{array}$$

However, in reality, the scattering lengths are large:

$$a_{1S_0} = -23.714 \text{ fm} \sim -16.6 M_\pi^{-1}, \quad a_{3S_1} = 5.42 \text{ fm} \sim 3.8 M_\pi^{-1}$$

Thus, it seems more appropriate to count $a \sim p^{-1}$. This leads to the expansion:

$$\mathcal{A} = -\frac{4\pi}{m} \left[\underbrace{\frac{1}{a^{-1} + ip}}_{\text{order } p^{-1}} + \underbrace{\frac{rp^2}{2(a^{-1} + ip)^2}}_{\text{order } p^0} + \underbrace{\frac{r^2p^4}{4(a^{-1} + ip)^3}}_{\text{order } p} + \dots \right]$$

Pionless EFT

The large scattering length signals non-perturbative physics. In order to accommodate for it, some fine tuning must be built in to the EFT.

The resulting power counting depends on the choice of renormalization conditions!

[for details see EE, Gegelia, Meißner, Nucl. Phys. B925 (2017) 161]

Consider a general expansion for the potential: $V = V^{\text{LO}} + V^{\text{NLO}} + V^{\text{N}^2\text{LO}} + \dots$

Want to assign powers of p to match: $\mathcal{A} = \mathcal{A}^{(-1)} + \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \dots$

LS equation: $\hat{\mathcal{A}}^{(-1)} = \hat{V}^{\text{LO}} - \hat{V}^{\text{LO}} \hat{G}_0 \hat{\mathcal{A}}^{(-1)} \rightarrow 1 + \hat{V}^{\text{LO}} \hat{G}_0 = \hat{V}^{\text{LO}} [\hat{\mathcal{A}}^{(-1)}]^{-1}$

Let: $\hat{V}^{\text{LO}} \sim \mathcal{O}(p^x) \rightarrow 1 + \hat{V}^{\text{LO}} \hat{G}_0 \sim \mathcal{O}(p^{1+x}) \rightarrow \begin{cases} \hat{G}_0 \sim \mathcal{O}(p), & x \leq -1 \\ \hat{G}_0 \sim \mathcal{O}(p^{-x}), & x > -1 \end{cases}$

A desired scaling of $\hat{G}_0$ can be realized by choosing the renormalization conditions:

• **Weinberg:** $\mu \sim \mathcal{O}(1), \mu_i \sim \mathcal{O}(p) \rightarrow x = 0 \rightarrow V_{\text{Weinberg}}^{\text{LO}} \sim \mathcal{O}(1)$

• **KSW:** $\mu, \mu_i \sim \mathcal{O}(p) \rightarrow x = -1 \rightarrow V_{\text{KSW}}^{\text{LO}} \sim \mathcal{O}(p^{-1})$

Scaling of V^{NLO} can be read off from:

$\hat{\mathcal{A}}^{(0)} = \hat{V}^{\text{NLO}} - \hat{V}^{\text{NLO}} \hat{G}_0 \hat{\mathcal{A}}^{(-1)} - \hat{\mathcal{A}}^{(-1)} \hat{G}_0 \hat{V}^{\text{NLO}} + \hat{\mathcal{A}}^{(-1)} \hat{G}_0 \hat{V}^{\text{NLO}} \hat{G}_0 \hat{\mathcal{A}}^{(-1)} \rightarrow \begin{cases} V_{\text{Weinberg}}^{\text{NLO}} \sim \mathcal{O}(p^2) \\ V_{\text{KSW}}^{\text{NLO}} \sim \mathcal{O}(1) \end{cases}$

Pionless EFT

Both choices of the renormalization conditions

- lead to self-consistent approaches
- are equivalent for pionless EFT but lead to different EFTs with pions
- involve some fine tuning beyond NDA [see: EE, Gegelia, Meißner, NPB 925 (2017) 161]

Leading order (p^{-1}):

$$\mathcal{A}^{(-1)} = \text{contact} - \text{loop} + \text{bubble} + \dots$$

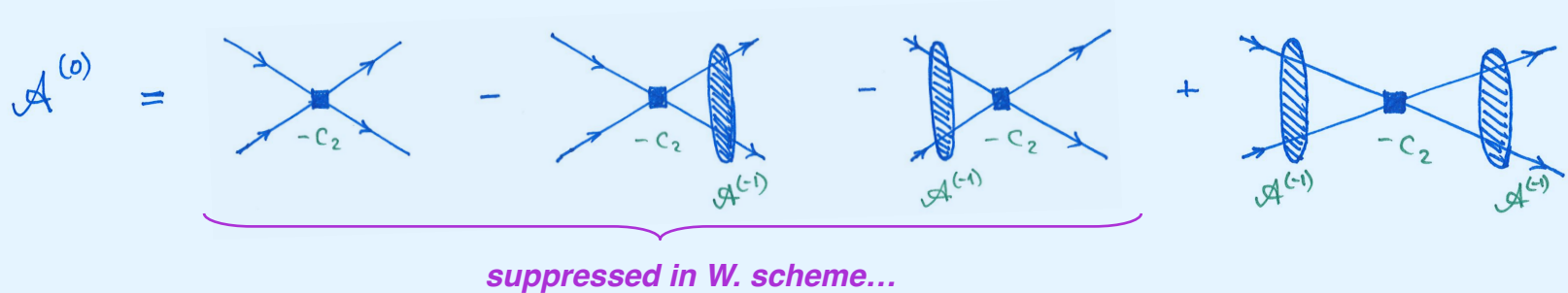
$$\begin{aligned} \mathcal{A}^{(-1)} &= -C_0 - C_0^2 I(p) - C_0^3 (I(p))^2 + \dots = -\frac{1}{C_0^{-1} - I(p)} = -\frac{1}{(C_0^R(\mu))^{-1} - I^R(\mu, p)} \\ &= -\frac{4\pi}{m} \frac{1}{\frac{4\pi}{m} (C_0^R(\mu))^{-1} + \frac{2}{\pi} \mu + ip} \stackrel{!}{=} -\frac{4\pi}{m} \frac{1}{a^{-1} + ip} \rightarrow C_0^R = \frac{4\pi}{m} \frac{1}{a^{-1} - \frac{2}{\pi} \mu} \end{aligned}$$

One recovers the Weinberg/KSW scaling of C_0^R depending on the choice of μ :

$$C_0^R \sim \mathcal{O}(1) \text{ for } \mu \sim \mathcal{O}(1); \quad C_0^R \sim \mathcal{O}(p^{-1}) \text{ for } \mu \sim \mathcal{O}(p).$$

Pionless EFT

Subleading order (p^0):



The subleading amplitude (including terms suppressed in W. scheme) reads:

$$\begin{aligned} \mathcal{A}^{(0)} &= -2C_2 p^2 - 2 \left(-C_2(p^2 I(p) + J_1(p)) \right) \mathcal{A}^{(-1)} - 2C_2 J_1(p) I(p) (\mathcal{A}^{(-1)})^2 \\ &= -I_3 + \underbrace{p^2 I(p)} \end{aligned}$$

For the sake of simplicity, choose $\mu_3 = 0$ (so that $I_3^R = 0$).

After renormalization ($I(p) \rightarrow I^R(\mu, p)$, $C_2 \rightarrow C_2^R(\mu)$), one finds:

$$\mathcal{A}^{(0)} = -2C_2^R p^2 \frac{(a^{-1} - \frac{2}{\pi}\mu)^2}{(a^{-1} + ip)^2} \stackrel{!}{=} -\frac{4\pi}{m} r p^2 \frac{1}{2(a^{-1} + ip)^2} \rightarrow C_2^R = \frac{\pi}{m} \frac{r}{(a^{-1} - \frac{2}{\pi}\mu)^2}$$

Again, we recover: $C_2^R \sim \mathcal{O}(1)$ for $\mu \sim \mathcal{O}(1)$; $C_2^R \sim \mathcal{O}(p^{-2})$ for $\mu \sim \mathcal{O}(p)$.

Pionless EFT: Implicit renormalization

In EFT with non-perturbative pions, the amplitude cannot be calculated analytically
⇒ renormalization has to be performed implicitly.

„The theory is fully specified by the values of the bare constants once a suitable regularization procedure is chosen. In principle, the renormalization program is straightforward: one calculates quantities of physical interest in terms of the bare parameters at given, large value of (ultraviolet cutoff) Λ . Once a sufficient number of physical quantities have been determined as functions of the bare parameters one inverts the result and expresses the bare parameters in terms of physical quantities, always working at some given, large value of Λ . Finally, one uses these expressions to eliminate the bare parameters in all other quantities of physical interest.“

Gasser, Leutwyler, Phys. Rep. **87** (1982) 77

Let's see how this works in pionless EFT...

Pionless EFT: Implicit renormalization

Define the contact potential; introduce a UV cutoff $\Lambda \sim \Lambda_b \sim M_\pi$; solve the LS equation; tune bare LECs $C_0(\Lambda)$, $C_2(\Lambda)$ to a , r .

For a sharp cutoff, one finds at NLO:

$$mC_0 = \frac{6\pi^2 (\beta - 6\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2})}{5\alpha\Lambda}, \quad mC_2 = \frac{6\pi^2 (\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2} - \alpha)}{\alpha\Lambda^3}$$

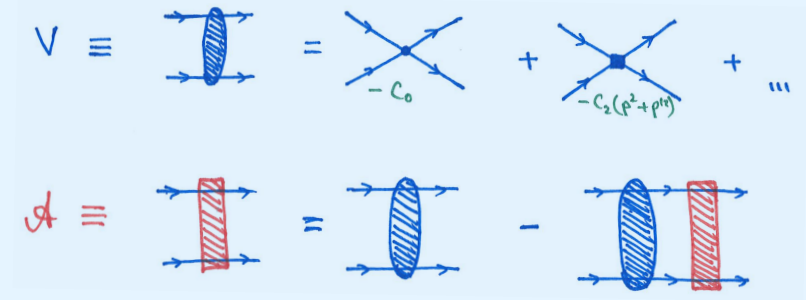
where I have introduced:

$$\alpha \equiv 16a^2\Lambda^2 - \pi a\Lambda (a\Lambda^2 r + 12) + 3\pi^2, \quad \beta \equiv 64a^2\Lambda^2 - \pi a\Lambda (3a\Lambda^2 r + 62) + 18\pi^2$$

Implicitly renormalized expression for the inverse amplitude:

$$\frac{4\pi}{m} \frac{1}{\mathcal{A}(p)} = \left[-\frac{1}{a} + \frac{1}{2}rp^2 + \frac{\pi(8 - 3a\Lambda^2 r(\pi\Lambda r - 8)) - 64a\Lambda}{12\pi\Lambda^3(\pi - 2a\Lambda)}p^4 + \mathcal{O}(p^6) \right] - ip$$

$\Rightarrow$ Well-defined and correct (up to higher-order terms) result for $\Lambda \sim r^{-1} \sim M_\pi$.
However, things may (and, in general, would) go wrong if choosing $\Lambda \gg r^{-1}$ (complex C_i 's, Wigner bound, peratization...)



Pionless EFT: Implicit renormalization

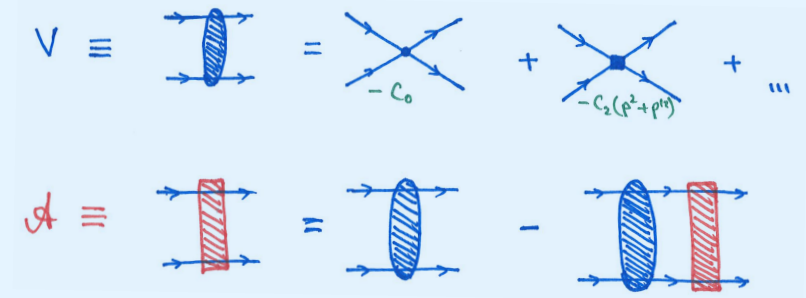
Define the contact potential; introduce a UV cutoff $\Lambda \sim \Lambda_b \sim M_\pi$; solve the LS equation; tune bare LECs $C_0(\Lambda)$, $C_2(\Lambda)$ to a , r .

For a sharp cutoff, one finds at NLO:

$$mC_0 = \frac{6\pi^2 (\beta - 6\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2})}{5\alpha\Lambda}, \quad mC_2 = \frac{6\pi^2 (\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2} - \alpha)}{\alpha\Lambda^3}$$

where I have introduced:

$$\alpha \equiv 16a^2\Lambda^2 - \pi a\Lambda (a\Lambda^2 r + 12) + 3\pi^2, \quad \beta \equiv 64a^2\Lambda^2 - \pi a\Lambda (3a\Lambda^2 r + 62) + 18\pi^2$$



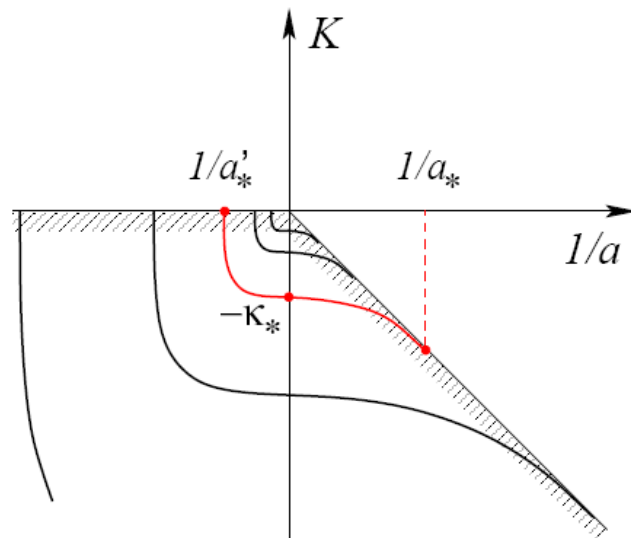
To summarize:

- Contrary to the previous cases, not all c.t. needed to remove UV divergences are included $\Rightarrow$ it is not legitimate to take the limit $\Lambda \rightarrow \infty$.
- Higher-order terms are indeed small provided $\Lambda \sim$ hard scale (NDA...).
- Implicit renormalization (i.e. no explicit splitting of C_i into $C_i^R(\mu)$ and $\Delta(\mu)$).
- Bare LECs $C_i(\Lambda)$ must be re-fitted at every order.

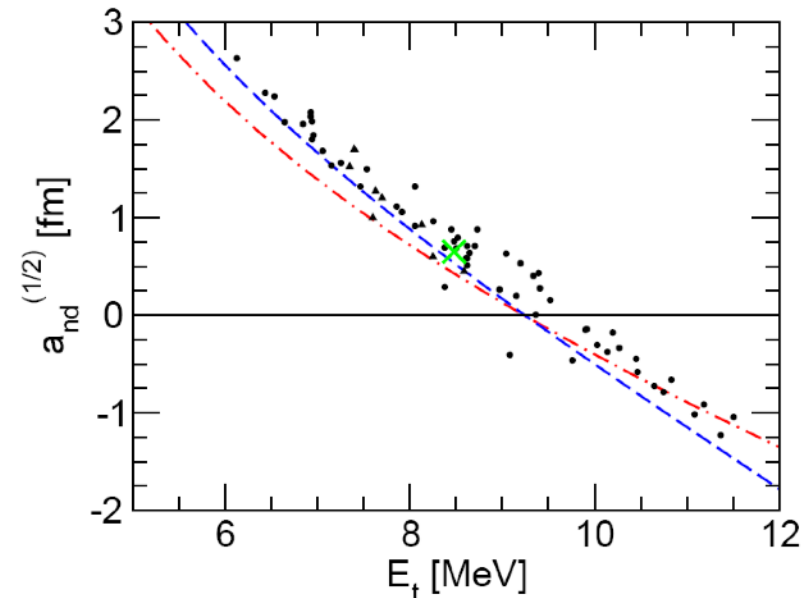
Pionless EFT: (some) applications

- Astrophysical reactions [Butler, Chen, Kong, Ravndal, Rupak, Savage, ...](#)
- Efimov physics and universality in few-body systems with large 2-body scatt. length (e.g. Phillips/Tjon „lines“) [Braaten, Hammer, Meißner, Platter, von Stecher, Schmidt, Moroz, ...](#)
- Halo-nuclei [Bedaque, Bertulani, Hammer, Higa, van Kolck, Phillips, ...](#)
- Parity violation [Schindler, Springer, Vanasse, ...](#) any many other topics...

Efimov effect (3-body spectrum)



Phillips line



Take-aways of part III

- the appearance of shallow NN states signals fine tuning (non-perturbative physics) that must be built in to an EFT
- different choices of renormalization conditions in pionless EFT lead to different power countings (KSW vs W)
- renormalization can also be carried out implicitly by tuning bare LECs to experimental data

IV: Inclusion of pions

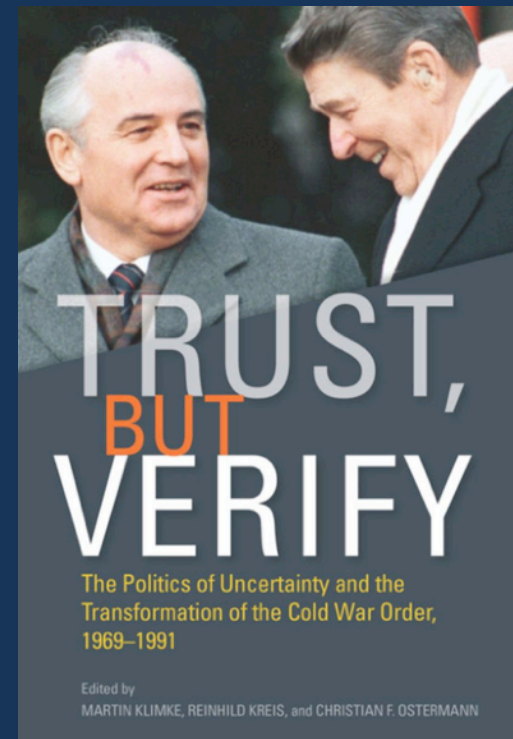
Are pions perturbative? How to test the long-range dynamics?
What is the predictive power of chiral EFT for few N's?

Outline

- Low-energy theorems (LETs) and the modified ERE
- KSW with perturbative pions
- Non-perturbative inclusion of pions

Further reading

van Haeringen, Kok, PRA 26 (1982) 1218
Lepage, *How to renormalize the Schrödinger equation*, nucl-th/9607029
Kaplan, Savage, Wise, NPB 534 (1998) 329
Cohen, Hansen, PRC 59 (1999) 13, 3047
Fleming, Mehen, Stewart, NPA 677 (2000) 313
EE, Gegelia, EPJA 41 (2009) 341
EE, *Nuclear forces from chiral EFT: A primer*, arXiv:1001.3229 [nucl-th]



Modified Effective Range Expansion (MERE)

LETs and the MERE

What are the low-energy theorems?

Two-range potential: $V(r) = V_L(r) + V_S(r)$

with $M_L^{-1} \gg M_H^{-1}$

- $F_l(k^2)$ is meromorphic in $|k| < M_L/2$

- $$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot [\delta_l(k) - \delta_l^L(k)]$$

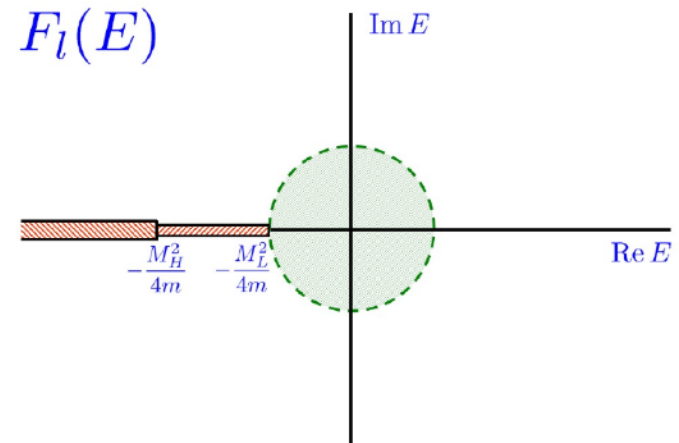
$$\underbrace{f_l^L(k)}_{\text{Jost function for } V_L(r)} = \lim_{r \rightarrow 0} \left(\frac{l!}{(2l)!} (-2ikr)^l \underbrace{f_l^L(k, r)}_{\text{Jost solution for } V_L(r)} \right)$$

Jost function for $V_L(r)$

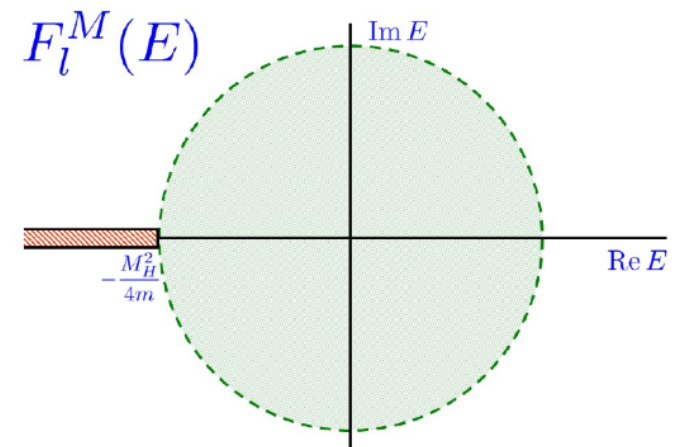
Jost solution for $V_L(r)$

$$M_l^L(k) = \text{Re} \left[\frac{(-ik/2)^l}{l!} \lim_{r \rightarrow 0} \left(\frac{d^{2l+1}}{dr^{2l+1}} \frac{r^l f_l^L(k, r)}{f_l^L(k)} \right) \right]$$

Per construction, F_l^M reduces to F_l for $V_L = 0$ and is meromorphic in $|k| < M_H/2$



← modified effective range function
van Haeringen, Kok '82



MERE and low-energy theorems

Example: proton-proton scattering

$$F_C(k^2) = C_0^2(\eta) k \cot[\delta(k) - \delta^C(k)] + 2k\eta h(\eta) = -\frac{1}{a^M} + \frac{1}{2}r^M k^2 + v_2^M k^4 + \dots$$

where $\underbrace{\delta^C \equiv \arg \Gamma(1 + i\eta)}_{\text{Coulomb phase shift}}, \quad \eta = \frac{m}{2k}\alpha, \quad \underbrace{C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}}_{\text{Sommerfeld factor}}, \quad h(\eta) = \text{Re}\left[\underbrace{\Psi(i\eta)}_{\text{Digamma function}}\right] - \ln(\eta)$
 $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$

MERE and low-energy theorems

Long-range forces impose correlations between the ER coefficients (low-energy theorems)
 Cohen, Hansen '99; Steele, Furnstahl '00

The emergence of the LETs can be understood in the framework of MERE:

$$\underbrace{F_l^M(k^2)}_{\substack{\text{meromorphic for} \\ k^2 < (M_H/2)^2}} \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot [\delta_l(k) - \delta_l^L(k)]$$

can be computed if the long-range force is known

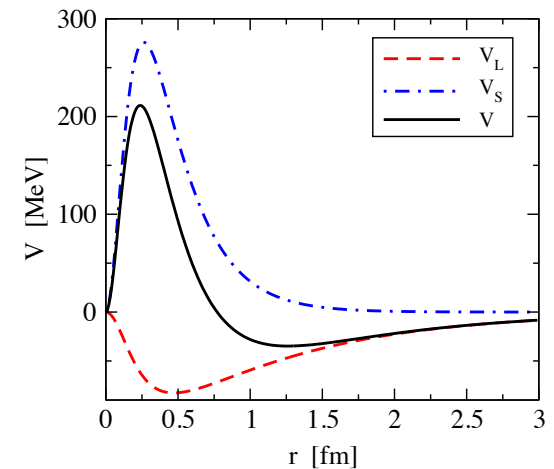
- approximate $F_l^M(k^2)$ by first 1,2,3,... terms in the Taylor expansion in k^2
- calculate all “soft” quantities
- reconstruct $\delta_l^L(k)$ and predict all coefficients in the ERE

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} + \underbrace{v_H e^{-M_H r} f(r)}_{V_H}$$

where $f(r) = \frac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0$, $v_L = -0.875$, $M_H = 3.75$, $v_H = 7.5$ (all in fm^{-1})



ERE and MERE

	a	r	v_2	v_3	v_4
$F_0 [\text{fm}^n]$	5.458	2.432	0.113	0.515	-0.993
$F_0^M [M_S^{-n}]$	1.710	-1.063	-0.434	-0.680	2.624

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r}}_{V_L} f(r) +$$

where $f(r) = \frac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0$, $v_L = -0.875$,

ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm ⁿ]	5.458	2.432	0.113	0.515	-0.993
F_0^M [M_S^{-n}]	1.710	-1.063	-0.434	-0.680	2.624

for an analytic example, see EE, Gegelia, EPJ A41 (2009) 341

Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r	2.447(38)	2.432197161	2.432197161	2.432197161
v_2	0.12(11)	0.1132(29)	0.112815751	0.112815751
v_3	0.61(12)	0.517(16)	0.51533(20)	0.51529
v_4	-0.95(5)	-0.991(14)	-0.9925(11)	-0.9928

Chiral EFT for NN scattering

KSW with perturbative pions

Recall the differences between the W and KSW counting schemes:

- **Weinberg:** $\mu \sim \mathcal{O}(1), \mu_i \sim \mathcal{O}(p) \rightarrow V_{\text{Weinberg}}^{\text{LO}} \sim \mathcal{O}(1), V_{\text{Weinberg}}^{\text{NLO}} \sim \mathcal{O}(p^2)$
[i.e. scaling of C_{2n} according to NDA ($\sim \mathcal{O}(1)$)]
- **KSW:** $\mu, \mu_i \sim \mathcal{O}(p) \rightarrow V_{\text{KSW}}^{\text{LO}} \sim \mathcal{O}(p^{-1}), V_{\text{KSW}}^{\text{NLO}} \sim \mathcal{O}(1)$
[i.e. scaling of C_{2n} as $C_{2n} \sim \mathcal{O}(p^{-1-n})$]

While the two schemes are equivalent for pionless theory, they suggest different scenarios for pionful (chiral) EFT:

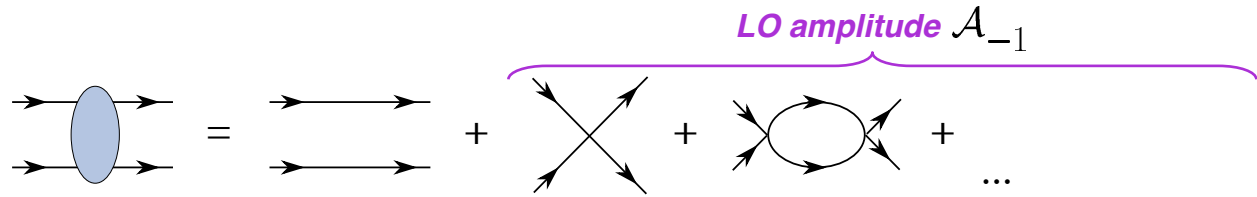
$$V_{1\pi} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + M_\pi^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \sim \mathcal{O}(1)$$

OPE is expected to be:

- LO contribution (nonperturbative) in the Weinberg scheme,
- NLO contribution (perturbative) in the KSW scheme.

Chiral EFT for NN: The KSW approach

- Leading order:



- NLO:

$$\mathcal{A}_0 = \underbrace{\text{Diagram I}}_{\mathcal{A}_0^{(I)}} + \underbrace{\text{Diagram II}}_{\mathcal{A}_0^{(II)}} + 2 \underbrace{\text{Diagram III}}_{\mathcal{A}_0^{(III)}} + \underbrace{\text{Diagram IV}}_{\mathcal{A}_0^{(IV)}} + \underbrace{\text{Diagram V}}_{\mathcal{A}_0^{(V)}}$$

$$\mathcal{A}_0^{(I)} = -C_2^{(1S_0)} p^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(1S_0)}} \right]^2, \quad \mathcal{A}_0^{(II)} = \left(\frac{g_A^2}{2f^2} \right) \left(-1 + \frac{m_\pi^2}{4p^2} \ln \left(1 + \frac{4p^2}{m_\pi^2} \right) \right)$$

$$\mathcal{A}_0^{(III)} = \frac{g_A^2}{f^2} \left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi} \right) \left(-\frac{(\mu + ip)}{m_\pi} + \frac{m_\pi}{2p} \left[\tan^{-1} \left(\frac{2p}{m_\pi} \right) + \frac{i}{2} \ln \left(1 + \frac{4p^2}{m_\pi^2} \right) \right] \right)$$

$$\mathcal{A}_0^{(IV)} = \frac{g_A^2}{2f^2} \left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi} \right)^2 \left(-\left(\frac{\mu + ip}{m_\pi} \right)^2 + \left[i \tan^{-1} \left(\frac{2p}{m_\pi} \right) - \frac{1}{2} \ln \left(\frac{m_\pi^2 + 4p^2}{\mu^2} \right) + 1 \right] \right)$$

$$\mathcal{A}_0^{(V)} = -D_2^{(1S_0)} m_\pi^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(1S_0)}} \right]^2$$

For more details see:

Kaplan, Savage, Wise, Nucl. Phys. B534 (1998) 329

LETs for S-waves: KSW approach

Use these results to test the LETs for S-waves: [Cohen, Hansen, PRC 59 (1999) 13]

$$p \cot \delta_0(p) = \frac{4\pi}{m} \left[\frac{1}{\mathcal{A}_{-1}} - \frac{\mathcal{A}_0}{(\mathcal{A}_{-1})^2} + \dots \right] + ip \stackrel{!}{=} -\frac{1}{a} + \frac{1}{2}rp^2 + v_2p^4 + v_3p^6 + v_4p^8 + \dots$$

Express the LECs C_0, C_2 , in terms of a and r to predict the shape parameters, e.g.:

$$v_2 = \frac{g_A^2 m}{16\pi F_\pi^2} \left(-\frac{16}{3a^2 M_\pi^4} + \frac{32}{5a M_\pi^3} - \frac{2}{M_\pi^2} \right), \quad v_3 = \frac{g_A^2 m}{16\pi F_\pi^2} \left(\frac{16}{a^2 M_\pi^6} - \frac{128}{7a M_\pi^5} + \frac{16}{3M_\pi^4} \right), \dots$$

1S_0 partial wave	a [fm]	r [fm]	v_2 [fm ³]	v_3 [fm ⁵]	v_4 [fm ⁷]
NLO KSW Cohen, Hansen '99	fit	fit	-3.3	18	-108
Nijmegen PWA	-23.7	2.67	-0.5	4.0	-20

3S_1 partial wave	a [fm]	r [fm]	v_2 [fm ³]	v_3 [fm ⁵]	v_4 [fm ⁷]
NLO KSW Cohen, Hansen '99	fit	fit	-0.95	4.6	-25
Nijmegen PWA	5.42	1.75	0.04	0.67	-4.0

→ large deviations suggest that pions should be treated nonperturbatively...

[even stronger evidence comes from phase shifts at N²LO, see: Fleming, Mehen, Stewart, NPA 677 (2000) 313]

Nonperturbative inclusion of pions

LO scattering amplitude:

$$T(\vec{p}', \vec{p}) = \left[V_{\text{cont}}(\vec{p}', \vec{p}) + V_{1\pi}(\vec{p}', \vec{p}) \right] + m \int \frac{d^3 l}{(2\pi)^3} \frac{\left[V_{\text{cont}}(\vec{p}', \vec{l}) + V_{1\pi}(\vec{p}', \vec{l}) \right] T(\vec{l}, \vec{p})}{p^2 - l^2 + i\epsilon}$$

Complications (as compared to pionless theory):

- $V_{1\pi}$ is not separable, no analytic results beyond 2 loops are available,
- $1/r^3$ singularity of $V_{1\pi}$

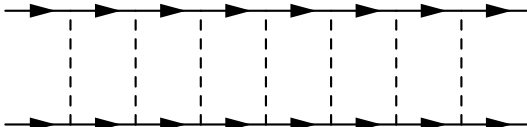
Static OPEP in coordinate space:

$$V_{1\pi}(\vec{r}) = \left(\frac{g_A}{2F_\pi} \right)^2 \tau_1 \cdot \tau_2 \left[M_\pi^2 \frac{e^{-M_\pi r}}{12\pi r} \left(\underbrace{S_{12}(\hat{r}) \left(1 + \frac{3}{M_\pi r} + \frac{3}{(M_\pi r)^2} \right)}_{\text{singular potential in all S=1 channels}} + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) - \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta^3(r) \right]$$

(solutions to the Schröd./LS equation still exist in repulsive cases)

⇒ Need ∞ many c.t.'s in all spin-triplet channel to remove UV divergences from iterations...

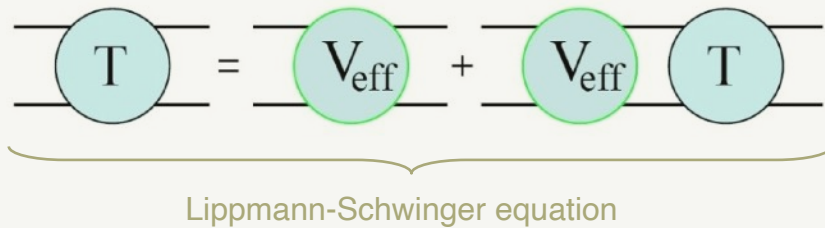
E.g.:



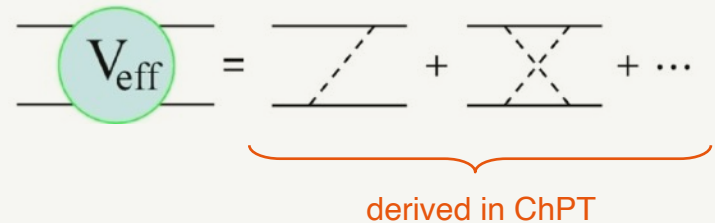
$$\propto \frac{1}{d-4} \vec{p}^6 m_N^6 \quad (\text{spin-triplet})$$

Chiral EFT for nuclear systems

Ladder graphs are responsible for the failure of perturbation theory and must be re-summed:



The diagram shows the Lippmann-Schwinger equation for the transition amplitude T . It is represented as a circle labeled T on the left, followed by an equals sign, then a circle labeled V_{eff} , followed by a plus sign, then another circle labeled V_{eff} followed by a circle labeled T . A bracket underneath the entire right-hand side is labeled "Lippmann-Schwinger equation".

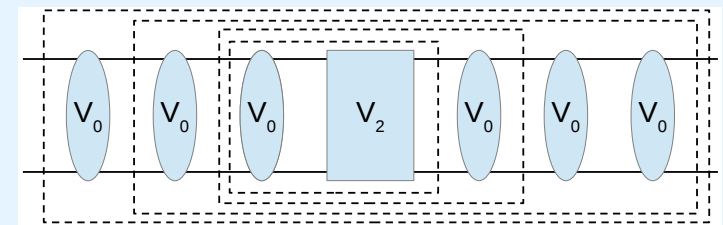


The diagram shows the expansion of the effective potential V_{eff} in Chiral Perturbation Theory (ChPT). It is represented as a circle labeled V_{eff} on the left, followed by an equals sign, then a series of diagrams: a single dashed line, a crossed dashed line, and an ellipsis. A bracket underneath the diagrams is labeled "derived in ChPT".

Nuclear forces and currents = irreducible parts of the amplitude (scheme-dependent)

Divergent integrals in the Lippmann-Schwinger equation are regularized using a cutoff Λ :

- the „RG invariant“ approach with $\Lambda \gg \Lambda_b$: $T \sim 1 + \Lambda + \Lambda^2 + \dots = (1 - \Lambda)^{-1}$ van Kolck, Long, Yang, ...
 - criticized in EE, Gegelia, EPJA 41 (09) 341; EE, Gasparyan, Gegelia, Meißner, EPJA 54 (18) 186
 - not cutoff-independent RG-invariant beyond LO Gasparyan, EE, PRC 107 (23) 034001
- finite- Λ EFT with $\Lambda \lesssim \Lambda_b \sim 600$ MeV Lepage, EE, Gegelia, Meißner, Reinert, Entem, Machleidt, ...
 - phenomenologically successful; approximate Λ -independence verified a posteriori
 - renormalizability (in the EFT sense) has been rigorously proven to NLO using the BPHZ subtraction method (forest formula) Gasparyan, EE, PRC 105 (22) 024001; PRC 107 (23) 044002



LETs for S-waves: KSW vs Weinberg

LETs for neutron-proton scattering: nonperturbative vs perturbative OPEP

	a [fm]	r [fm]	v_2 [fm ³]	v_3 [fm ⁵]	v_4 [fm ⁷]
¹S₀ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.50	−1.9	8.6(8)	−37(10)
NLO EE et al., EPJA51 (15) 71	fit	fit	−0.61 ... − 0.55	5.1 ... 5.5	−30.8 ... − 29.6
NLO KSW Cohen, Hansen '98	fit	fit	−3.3	18	−108
Empirical values	−23.7	2.67	−0.5	4.0	−20
³S₁ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.60	−0.05	0.82	−5.0
NLO Baru et al., PRC94 (16) 014001	fit	fit	0.06	0.70	−4.0
NLO KSW Cohen, Hansen '98	fit	fit	−0.95	4.6	−25
Empirical values	5.42	1.75	0.04	0.67	−4.0

- perturbative inclusion of pions (KSW approach) fails
- ¹S₀ channel: limited predictive power of the LETs due to the weakness of the OPEP; taking into account the range correction (NLO) leads to improvement
- ³S₁ channel: LETs work as advertised (strong tensor part of the OPEP)

Take-aways of part IV

- long-range interactions govern energy dependence of the amplitude and lead to correlations between coefficients in the ERE (LETs) that can be tested
- the failure of the LETs in the KSW approach suggests that pion exchange should be treated non-perturbatively
- iterations of the 1π -exchange are non-renormalizable (in the usual sense) $\Rightarrow$ finite-cutoff formulation of chiral EFT