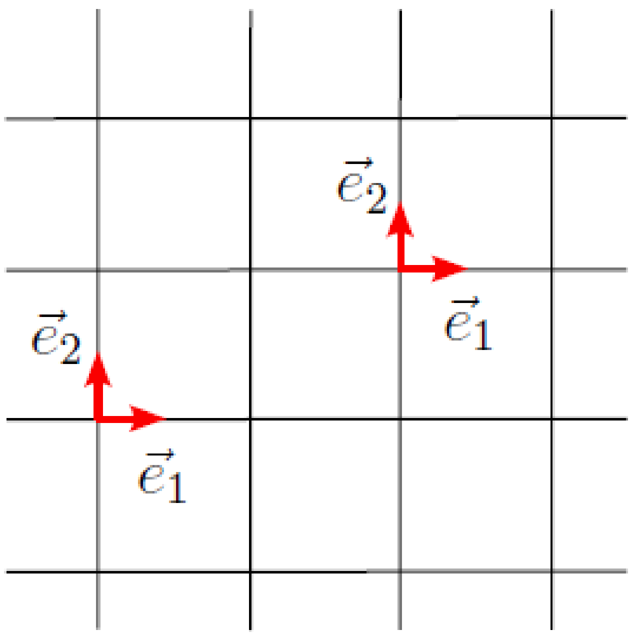
TECHNISCHE UNIVERSITÄT DARMSTADT

II. Fundamentals of Continuum Mechancis

Supplimentary slides

MECHANICS of FUNCTIONAL MATERIALS

2.1 Conventions and Theorem

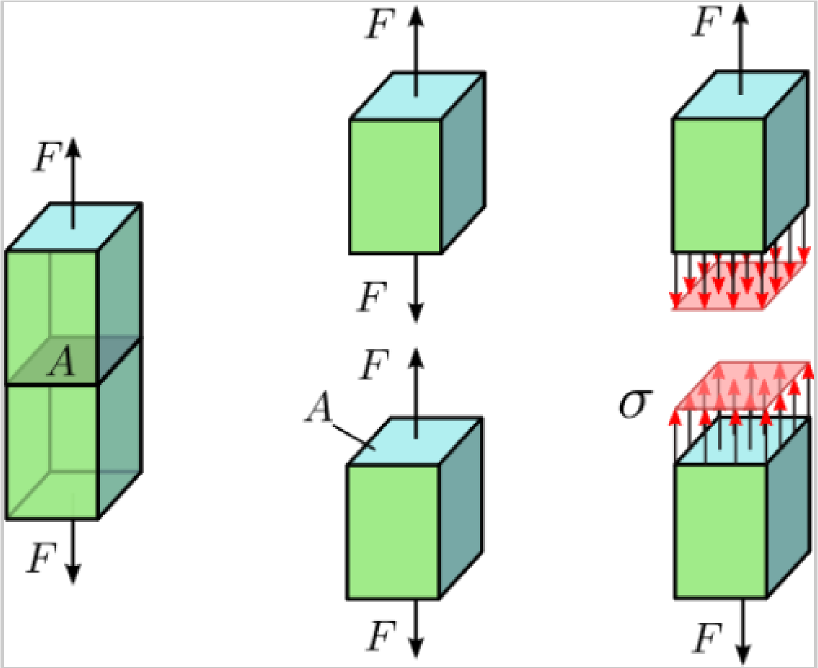


Cartesian coordinates

Please refer to the mansucript for more details. Feel free to ask questions in Q & A session in Zoom.

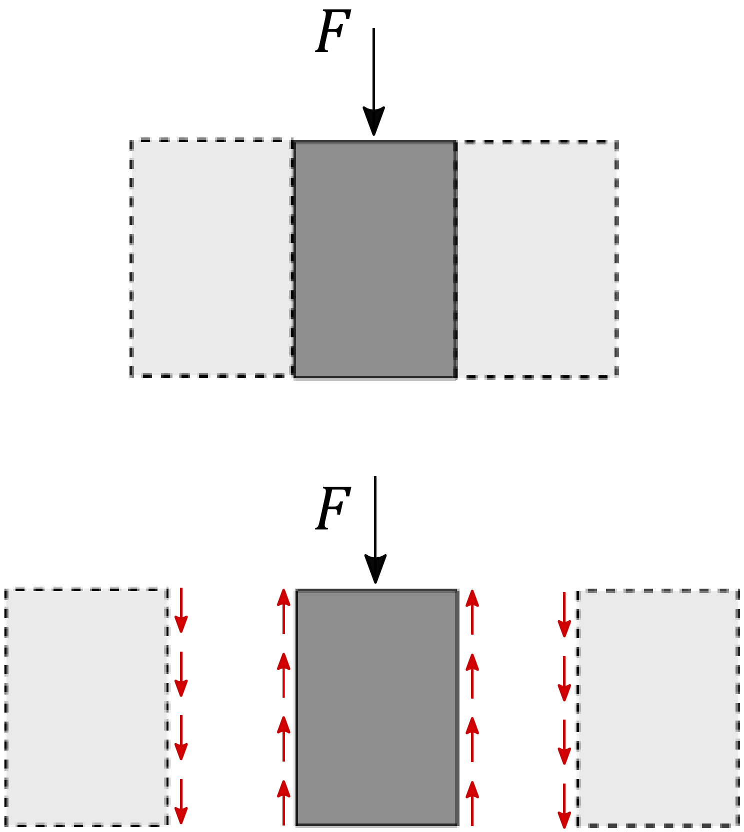
2.2 Stress
Stress vector

Normal stress

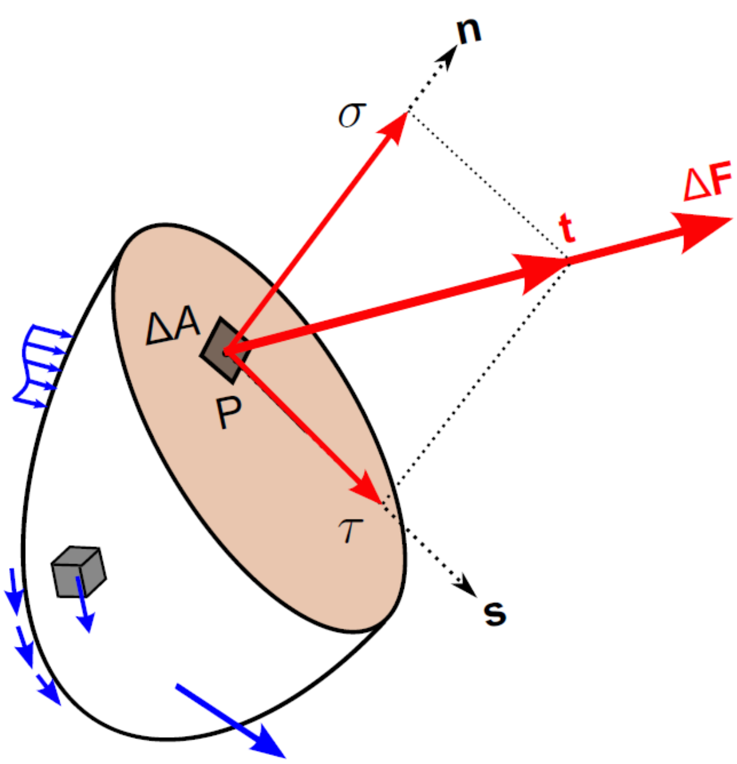


2.2 Stress
Stress vector

Shear stress



2.2 Stress
Stress vector



Stress vector at a material point:

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} = \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}A}$$

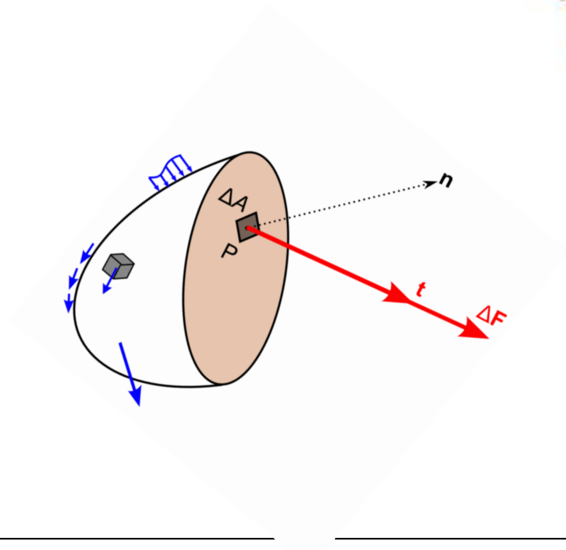
Decomposition of the stress vector:

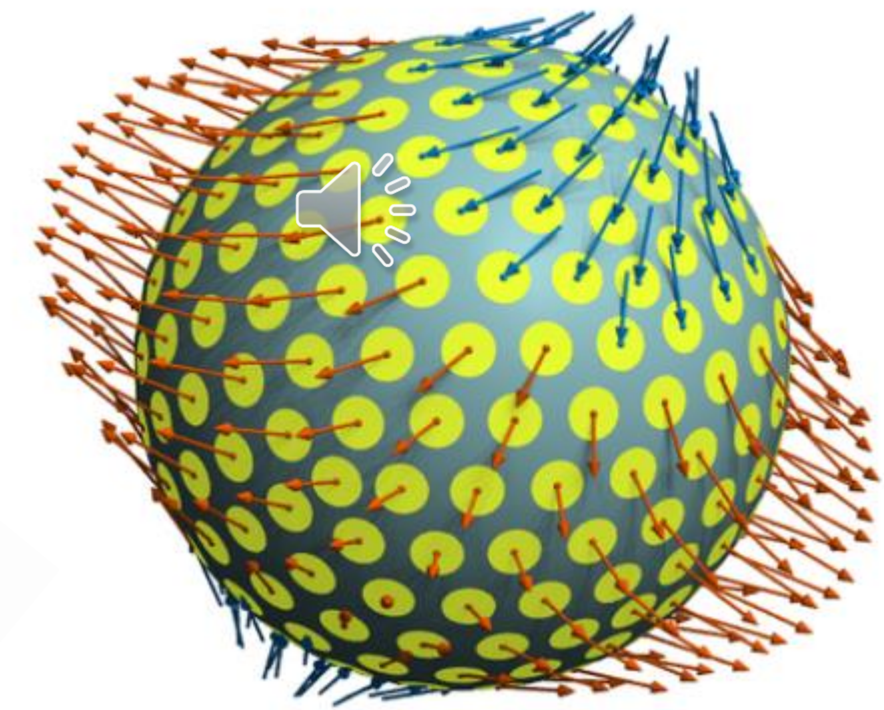
$$\text{normal stress}: \quad \sigma = \mathbf{t} \cdot \mathbf{n}$$

$$\text{shear stress}: \quad \tau = \sqrt{\mathbf{t} \cdot \mathbf{t} - \sigma^2}$$

2.2 Stress
Stress vector

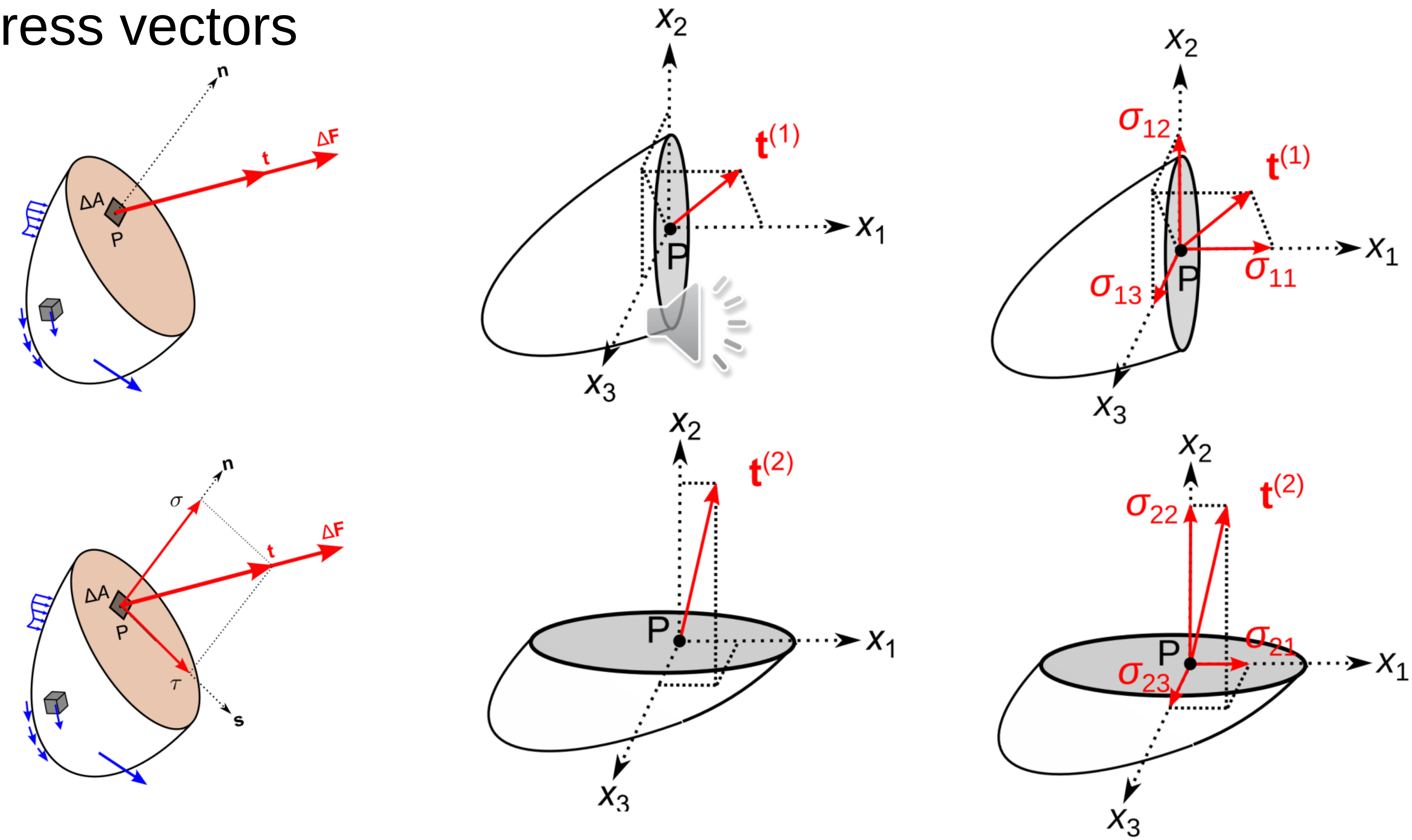
Stress vectors on different cross sections at one point





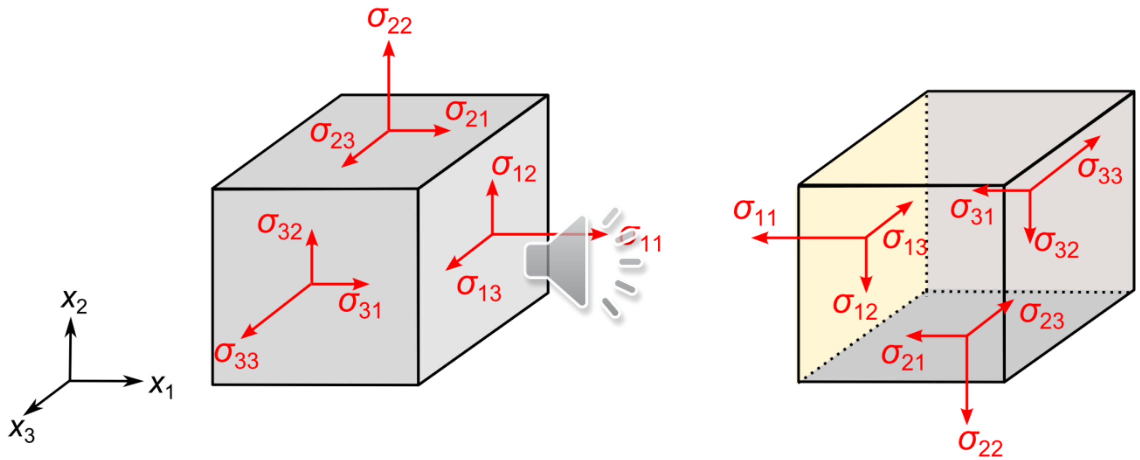
2.2 Stress
Stress tensor

Stress vectors



2.2 Stress
Stress tensor

Stress components in three perpenticular cross sections.



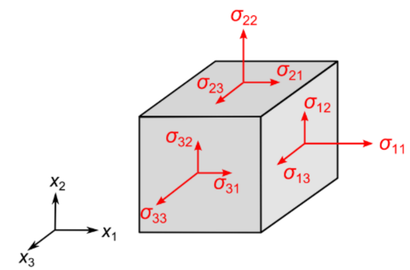
Alternatively we can write all the components in one matrix

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}_{x_i-Coor.}$$

2.2 Stress
Stress tensor

Comments

- **Double indices notation**: the first index describes the direction of the normal vector of the cross section, while the second indicates the direction of the stress component itself.



- **Symmetry of the stress tensor**, and thus also the symmetry of the matrix: This is due to the momentum balance, the shear components in two perpenticular sections.

$$\sigma_{21} = \sigma_{12}, \quad \sigma_{23} = \sigma_{32}, \quad \sigma_{31} = \sigma_{13}$$

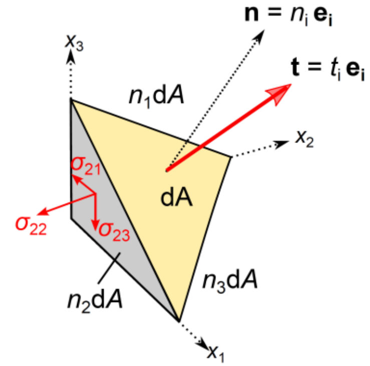
$$\sigma_{ij} = \sigma_{ji}$$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ sym & & \sigma_{33} \end{bmatrix}_{x_i-Coor.}$$

2.2 Stress
Stress tensor

- **Cauchy's formula**

$$t_i = \sigma_{ij} n_j = \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3 = \sum_{j=1}^{3} \sigma_{ij} n_j$$



$$v'_i = a_{ij} v_j$$
$$T' = T$$

- **Transformation relation**

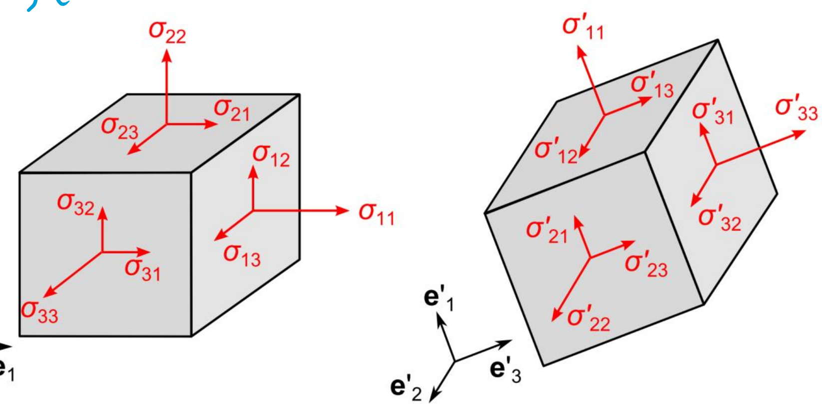
$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$$

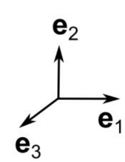
$$\sigma'_{13} = a_{1k} a_{3l} \sigma_{kl}$$

where a_{ik} is the rotation tensor between the two coordinates.

$$a_{ik} = \mathbf{e}'_i \cdot \mathbf{e}_k = \cos(x'_i, x_k)$$

It appears twice in the transformation relation. Thus the stress tensor is of 2nd order.





$$[\sigma_{ij}]_{x_i\text{-coor}} = [\sigma'_{ij}]_{x'_i\text{-coor.}}$$

2.2 Stress
Stress tensor

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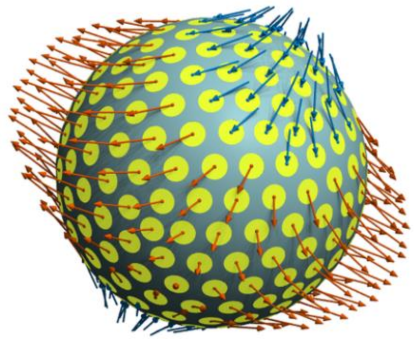
- **Principal stress** $\sigma_1, \sigma_2, \sigma_3$: Extreme values of the normal stress wrt. the rotation of coordinates. There exists always a special coordinate system, in which the shear stress components vanish and there left only with normal stress components. This coordinate system is called principle coordinates, and the related normal stress components are called principle stresses.

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}_{x_i-Coor.} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}_{\text{Principle-axises}}$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$



Likewise there is a particular coordinate, in which the shear stress take extreme values, the so-called principle shear stress τ_{max}. The related normal stresses are in general not zero, but the mean of the principle stresses.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

The related normal stresses are in general not zero, but the mean of the principle stresses.

2.2 Stress
Stress tensor

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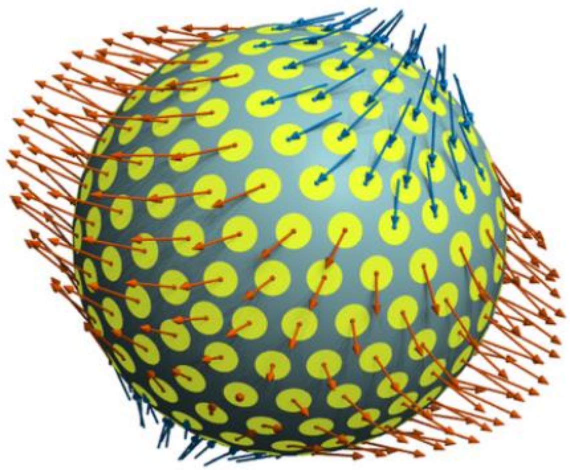

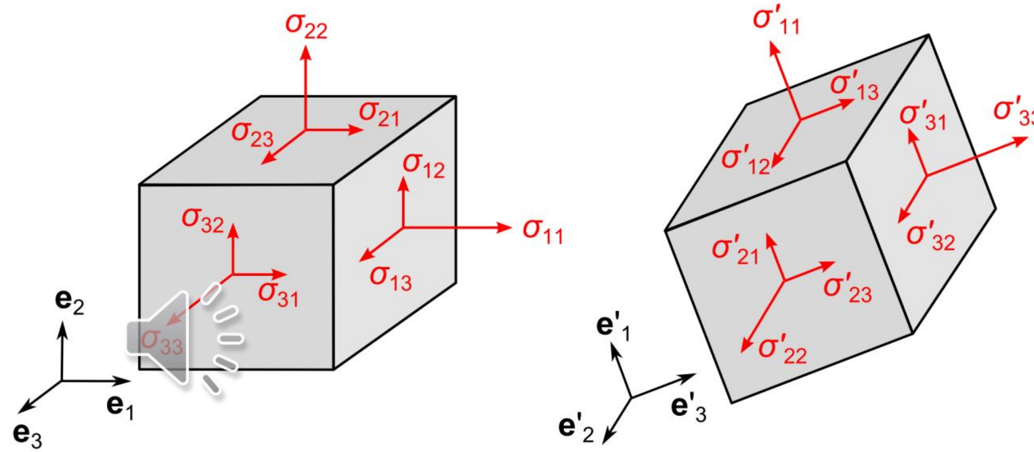

Invariants: quantities which do not change during rotation of coordinates

$$I_\sigma = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3$$

$$II_\sigma = \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$III_\sigma = \det\sigma_{ij} = \sigma_1\sigma_2\sigma_3$$

2.2 Stress
Stress tensor

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- **Decomposition** of the stress state into the hydrostatic state and the deviatoric state

$$\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \begin{bmatrix}\sigma_m & 0 & 0\\ 0 & \sigma_m & 0\\ 0 & 0 & \sigma_m\end{bmatrix} + \begin{bmatrix}\sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m\end{bmatrix}$$

hydrostatic deviatoric

$$\sigma_m = (\sigma_{11}+\sigma_{22}+\sigma_{33})/3$$

$$\sigma_{ij} = \sigma_m \delta_{ij} + s_{ij}$$

$$s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

The second invariant of the stress deviator:

$$J_2 = II_s = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

2.2 Stress
Stress tensor

- Alternative representation of stress components
  - $x_1, x_2, x_3 \rightarrow x, y, z$; Thus $\sigma_{11} \rightarrow \sigma_{xx},\ \sigma_{12} \rightarrow \sigma_{xy},\ \sigma_{13} \rightarrow \sigma_{xz}$
  - In this case, the normal stress $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are written in a short form of $\sigma_x, \sigma_y, \sigma_z$
  - $\sigma_{12}, \sigma_{23}, \sigma_{31} \quad \rightarrow \quad \tau_{12}, \tau_{23}, \tau_{31}$
  - Voigt Notation:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ sym & & \sigma_{33} \end{bmatrix}_{x_i-Coor.} \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}_{x_i-Coor.} \rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}_{x_i-Coor.}$$

$$11 \rightarrow 1 \quad 22 \rightarrow 2 \quad 33 \rightarrow 3 \quad 23 \rightarrow 4 \quad 31 \rightarrow 5 \quad 12 \rightarrow 6$$

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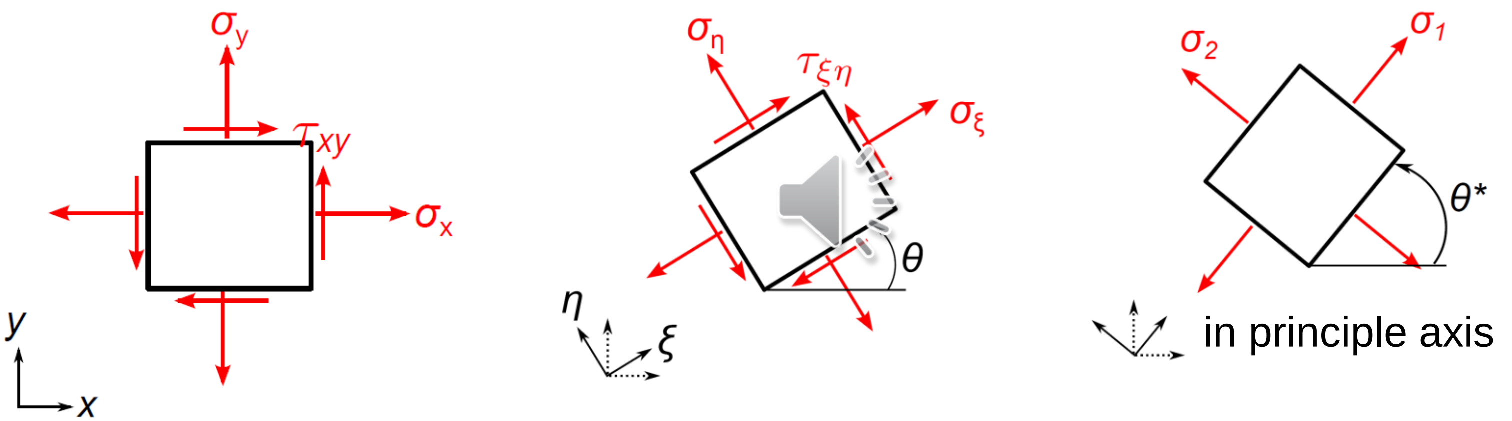
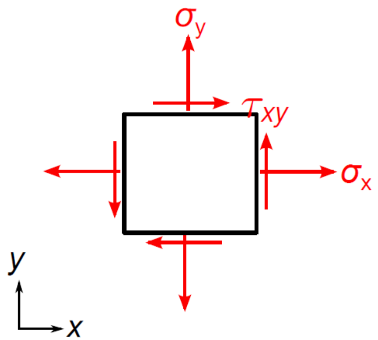
The stress tensor in 2D and its analysis:

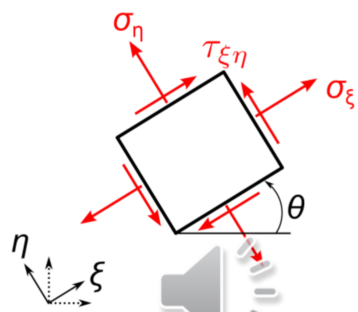


$$[\sigma_{ij}] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}_{xy-co.} = \begin{bmatrix} \sigma_\xi & \tau_{\xi\eta} \\ \tau_{\eta\xi} & \sigma_\eta \end{bmatrix}_{\xi\eta-co.} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}_{principle\ co.}$$

2.2 Stress
Stress tensor

The stress tensor in 2D and its analysis:





$$a_{ik} = \cos(\mathbf{e}'_i \cdot \mathbf{e}_k)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$$

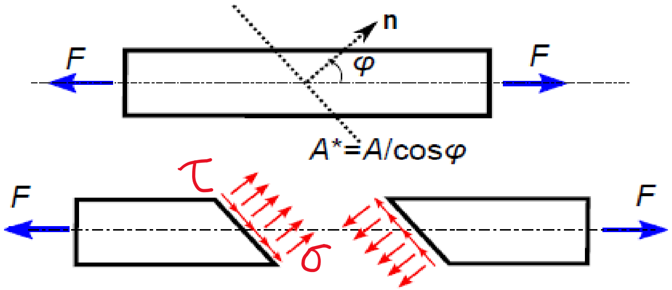
$$\begin{bmatrix} \sigma_\xi, \tau_{\xi\eta} \\ \tau_{\xi\eta}, \sigma_\eta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_x, \tau_{xy} \\ \tau_{xy}, \sigma_y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$\sigma_\xi = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} sin\theta \cos\theta$$

$$\sigma_\eta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_{\xi\eta} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

2.2 Stress
Stress analysis



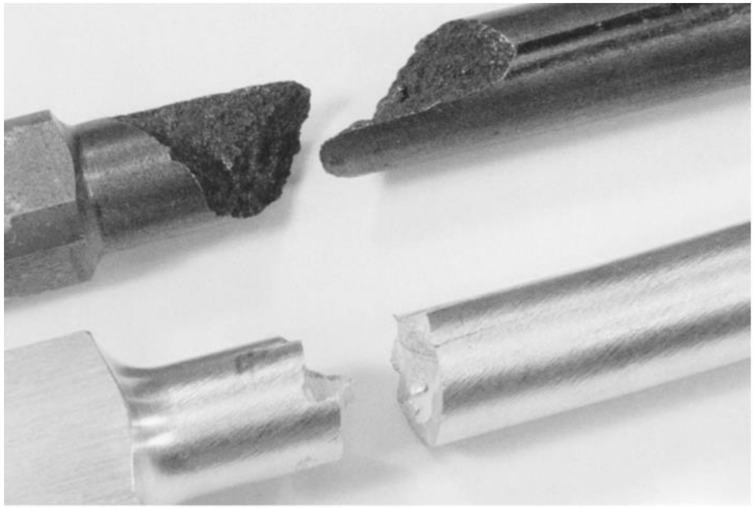
$$t = \frac{F}{A^*} = \frac{F}{A}\cos\varphi$$

$$\sigma = t\cos\varphi = \frac{F}{A}\cos^2\varphi$$

$$\tau = t\sin\varphi = \frac{F}{A}\cos\varphi\sin\varphi$$

$\sigma_{max} = \frac{F}{A}$, at $\varphi = 0$ → σ_{max}

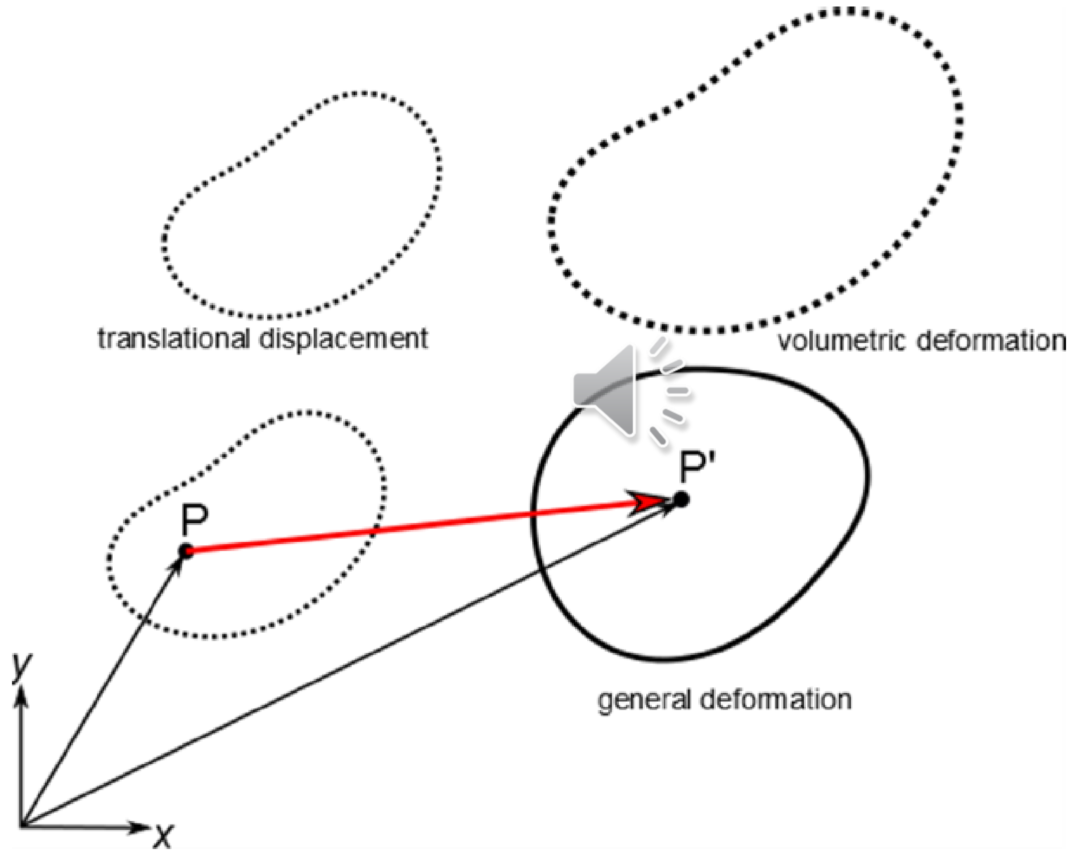
$\tau_{max} = \frac{F}{2A}$, at $\varphi = 45°$ → $\sigma = \frac{F}{2A}$, τ_{max}



Uniaxial tension test:
- Brittle materials fracture along vertical cross section
- Ductile materials fracture along 45 degree

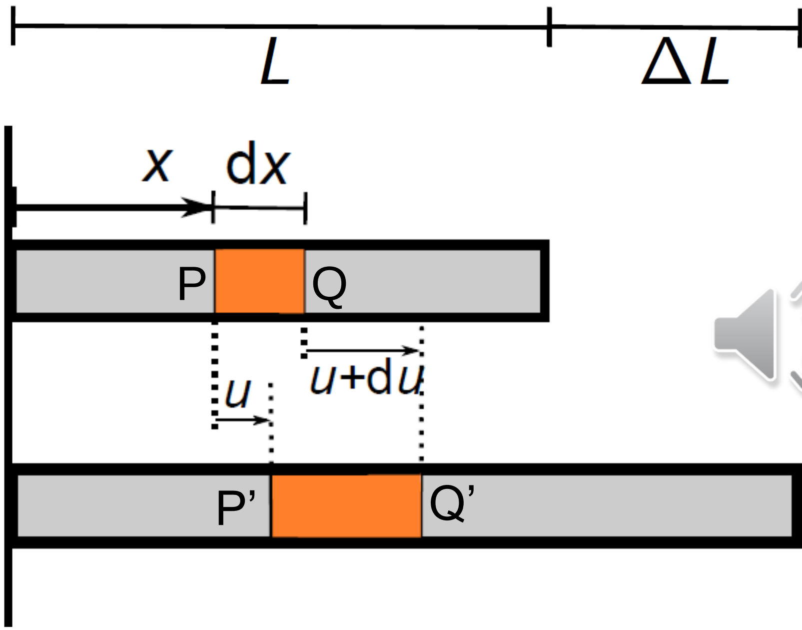
Torsion test
- Brittle materials fracture along 45 degree
- Ductile material fracture along vertical cross section

2.3 Kinematics (Deformation)
Displacement, Deformation



MECHANICS of FUNCTIONAL MATERIALS

2.3 Kinematics (Deformation)
Strain in 1D



Overall strain: $\varepsilon = \dfrac{\Delta L}{L}$

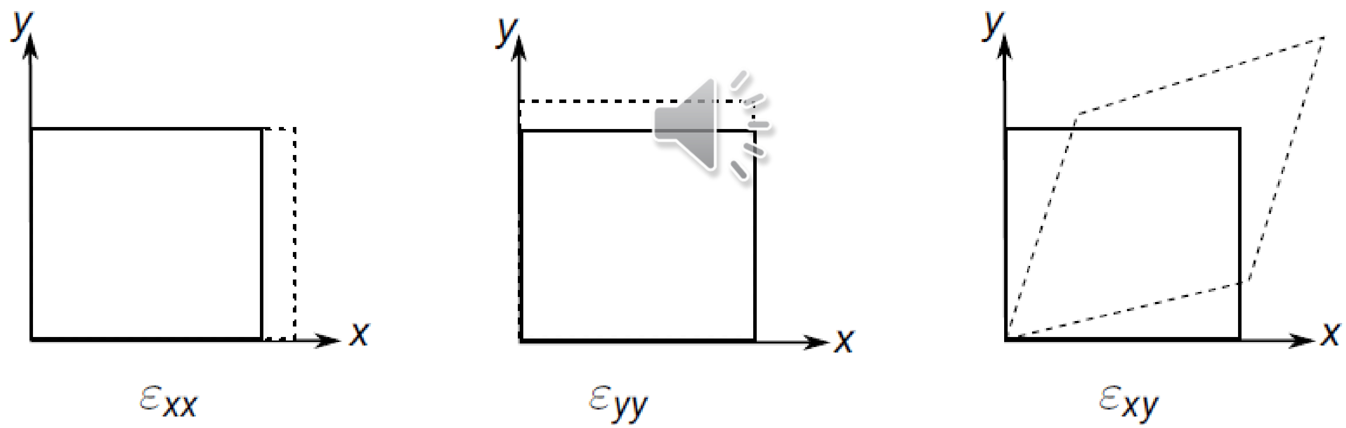
Displacement field $u(x)$

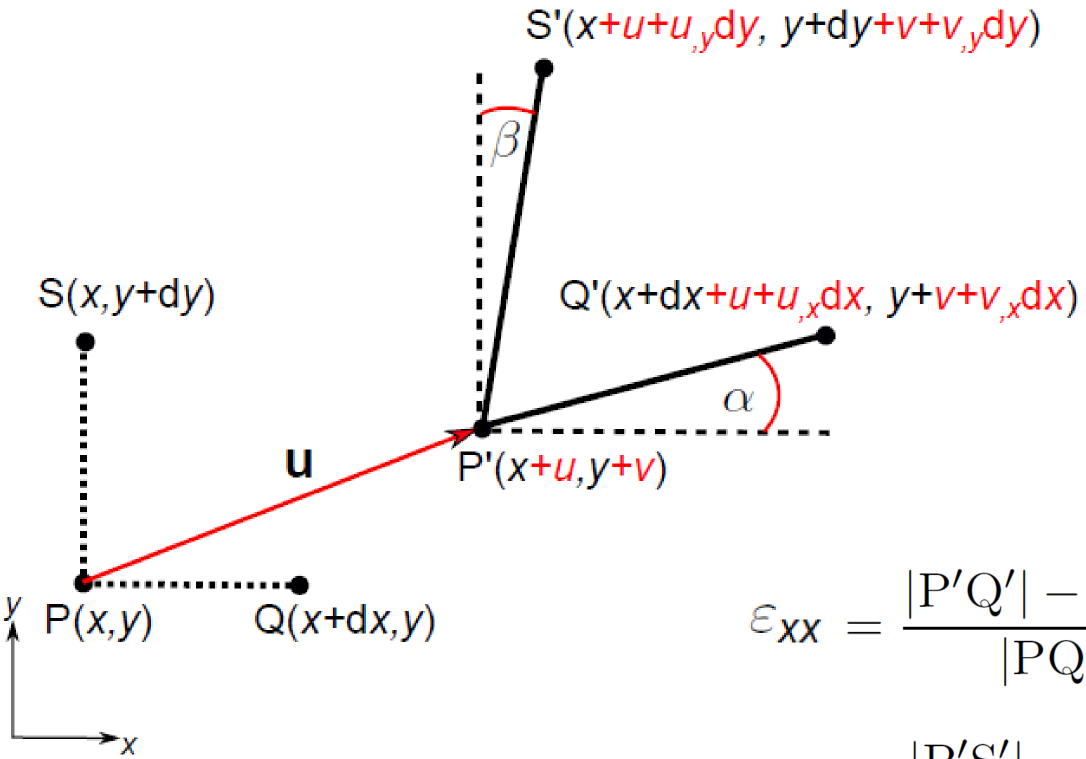
Local strain:

$$\varepsilon(x) = \lim_{dx \to 0} \frac{dx + du - dx}{dx} = \lim_{dx \to 0} \frac{dx + \frac{\partial u}{\partial x} dx - dx}{dx} = \frac{\partial u}{\partial x}$$

2.3 Kinematics (Deformation)
Strain tensor in 2D

We only consider the case of small deformation.





$$\varepsilon_{xx} = \frac{|P'Q'| - |PQ|}{|PQ|} = u_{,x} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{|P'S'| - |PS|}{|PS|} = v_{,y} = \frac{\partial v}{\partial y}$$

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ sym & \varepsilon_{yy} \end{bmatrix}$$

$$\varepsilon_{xy} = \frac{1}{2}(\alpha + \beta) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

2.3 Kinematics (Deformation)
strain tensor in 3D

Displacement field:

$u_1(x_1, x_2, x_3), \quad u_2(x_1, x_2, x_3), \quad u_3(x_1, x_2, x_3)$

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym & & \varepsilon_{33} \end{bmatrix}$$

$$\varepsilon_{ij} : \begin{cases} \text{normal strain} & i = j \\ \text{shear strain} & i \neq j \end{cases}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}), \quad \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}), \quad \varepsilon_{31} = \varepsilon_{13} = \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3})$$

or,

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Comments

- The strain tensor is also a tensor of 2nd order. The general features of a tensor 2nd order are also valid, including transformation relation, principle strains and invariants.

Transformation relation:

$$\varepsilon'_{ij} = a_{ik} a_{jl} \varepsilon_{kl}$$

In 2D:

$$\boxed{\begin{aligned}
\varepsilon_\xi &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \varepsilon_{xy}\sin 2\theta \\
\varepsilon_\eta &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) - \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta - \varepsilon_{xy}\sin 2\theta \\
\varepsilon_{\xi\eta} &= \tfrac{1}{2}\gamma_{\xi\eta} = \quad -\tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \varepsilon_{xy}\cos 2\theta
\end{aligned}}$$

where ε_ξ, ε_η and $\varepsilon_{\xi\eta} = \frac{1}{2}\gamma_{\xi\eta}$ are componentes in the new coordinate (ξ, η), which has a rotation angle θ to the cooerdinate system (x, y) .

- Decomposition into a volumetric part $\epsilon_{kk}\delta_{ij}/3$ and a deviatoric part e_{ij}:

$$\varepsilon_{ij} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} + e_{ij}$$

or

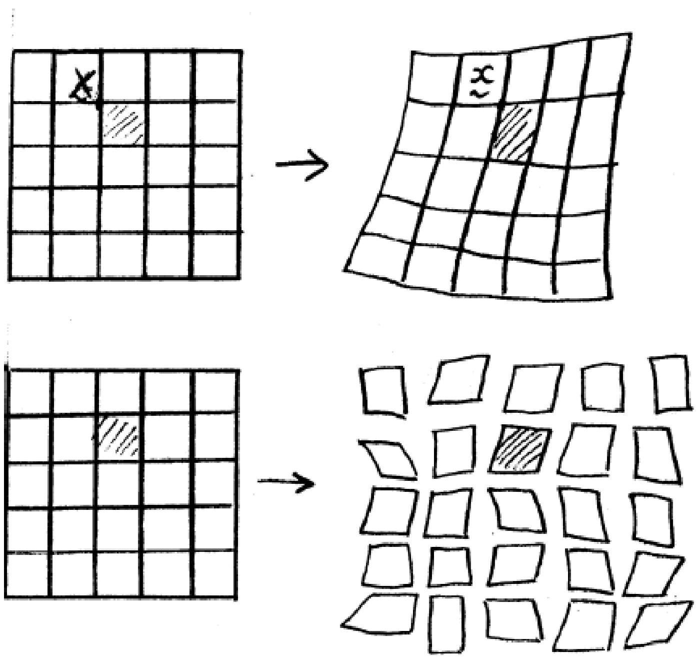
$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_{11}-\varepsilon_m & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22}-\varepsilon_m & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}-\varepsilon_m \end{bmatrix}$$

with

$$\varepsilon_m = \frac{1}{3}\varepsilon_{kk}$$

Volume strain : $\varepsilon_v = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$

- Compatibility condition: the displacement vector $\vec{u}$ has 3 components, but the strain tensor ε has 6 Components. It implies that the components of the strain tensor ε_{ij} are not independent from each other.



$$\begin{aligned} \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} &= 0, \quad \text{in 3D} \\ \varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} &= 0, \quad \text{in 2D} \end{aligned}$$

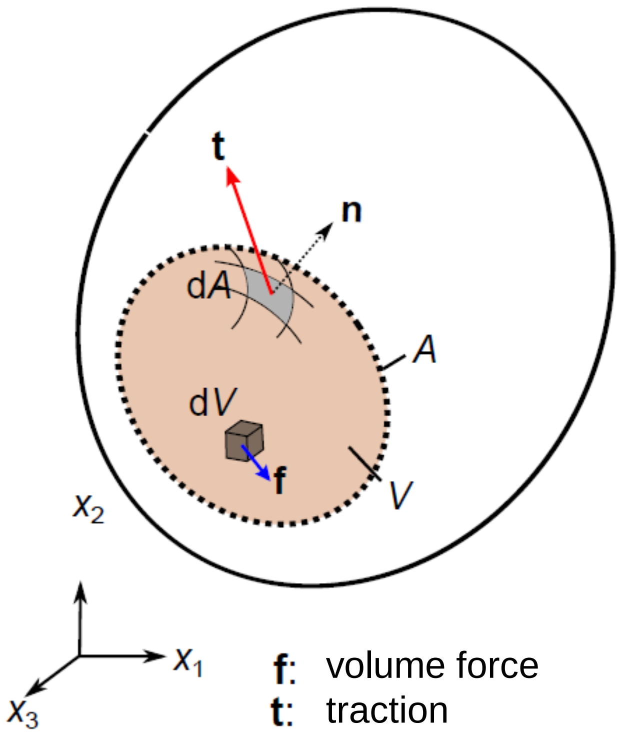
If the compatibility conditions are violated, there may exist no correspondingly smooth displacement field (rupture or interpenetrating of the materials)

- Alternative representation of the strains:
  + When the coordinates x, y, z are used, we have $\varepsilon_{11} = \varepsilon_{xx}, \varepsilon_{12} = \varepsilon_{xy}, \varepsilon_{13} = \varepsilon_{xz}$.
  + The normal stresses $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ are often written as $\varepsilon_x, \varepsilon_y, \varepsilon_z$.
  + $\gamma_{12} = 2\varepsilon_{12}, \gamma_{23} = 2\varepsilon_{23}, \gamma_{31} = 2\varepsilon_{31}$.
  + In the Voigt notation the strain components are saved in an array:

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym & & \varepsilon_{33} \end{bmatrix}_{x_i-Coor.} \rightarrow \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix}_{x_i-Coor.} \rightarrow \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ 2\varepsilon_{4} \\ 2\varepsilon_{5} \\ 2\varepsilon_{6} \end{bmatrix}_{x_i-Coor.}$$

2.4 Balance equation

Principle of linear momentum: the total force acting on a static or quasi-static volume should be balanced.



f: volume force
t: traction

Force acted in any volume V:

$$\int_A t_i \, \mathrm{dA} + \int_V f_i \, \mathrm{dV} = \int_A \sigma_{ji} n_j \, \mathrm{dA} + \int_V f_i \, \mathrm{dV} = 0$$

in which the Cauchy's Formula $t_i = \sigma_{ji} n_j$ is used. By using the Gauß theorem,

$$\int_V \left(\sigma_{ji,j} + f_i \right) \, \mathrm{dV} = 0$$

This equation holds for any volumen V. Thus,

$$\boxed{\sigma_{ji,j} + f_i = 0}$$

$$\boxed{\sigma_{ji,j} + f_i = 0}$$

or

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{13}}{\delta x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

The boundary conditions can be:

$$\sigma_{ji} n_j = t_i^* \qquad \text{or}$$

$$\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = t_1^*$$

$$\sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 = t_2^*$$

$$\sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = t_3^*$$

§ 2.5 Material law

material law

stress (loading) ⟷ strain (deformation)

What is material law?

- A material law is the physical relationship between the force quantities and the kinematic quantities.
- A material law depends on the material; it can only be characterized with the help of experiments or ab initio calculations.

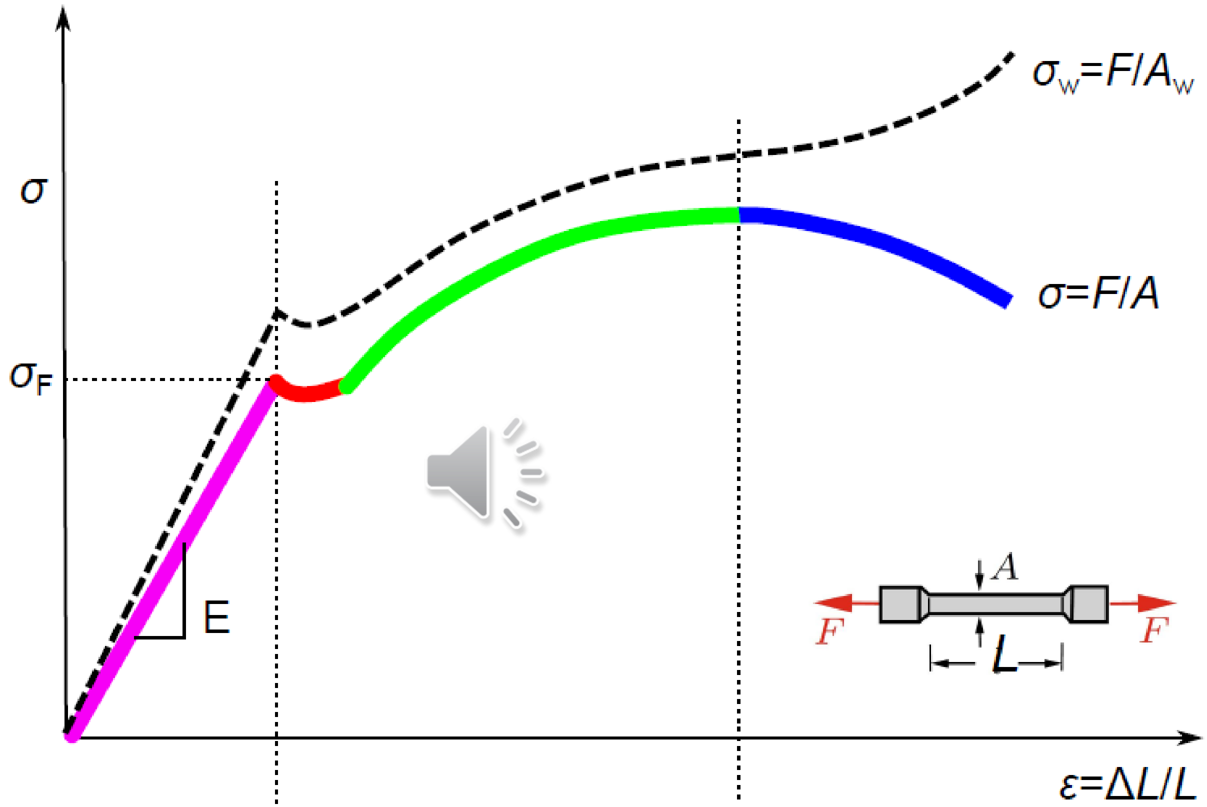
A typical stress-strain curve of steels

I. linear elastic

II. yielding

III. hardening

IV. necking



$E = \sigma/\varepsilon$: Young's modulus. σ_F: Yield strength
A_W: deformed cross section. σ_W: true stress

Linear elastic material law: 1D

$$\boxed{\sigma = E\,\varepsilon} \qquad \text{or} \qquad \boxed{\varepsilon = \frac{\sigma}{E}}$$

Comments:

- It is called the Hooke's law (Robert Hooke 1635 – 1703).
- The Young's modulus E is a material constant.
- E is usually the same for compressive and tensile loading.
- E has the same dimension as stress, $\mathrm{N/m^2}$ or Pa; $\mathrm{N/mm^2}$ or MPa; $\mathrm{kN/mm^2}$ or GPa

  At room temperature, for instance, E for steel is around 210 GPa, aluminum 70 GPa, and wood 7-20 GPa.

Linear elastic material law: 3D

Similar to the 1D case, we have the Hooke's in the 3D, by considering the stress and strain tensors.

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$$

where E_{ijkl} is the stiffness tensor.

One can reverse the Hooke's law and writes the strain components from the stress components.
The coefficient tensor is then called compliance tensor S_{ijkl}:

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

Both the stiffness and the compliance tensors are of the 4th order.

There should be in total $3^4 = 81$ constants for a general tensor of 4th order. However, E_{ijkl} has the following symmmetry features:

Due to the symmmetry of $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$,

$$E_{ijkl} = E_{jikl} = E_{ijlk} \qquad \text{(minor symmetry)}$$

Due to the quadratic form of the strain energy density for a linear elastic material:

$$E_{ijkl} = E_{klij} \qquad \text{(major symmetry)}$$

Therefore the independent number of components decreases from 81 to 21. In other words, a general anisotropic materials are expected to have maximal 21 elastic constants.

Similarly, the compliance tensor S_{ijkl} has both the minor and the major symmetries:

$$S_{ijkl} = S_{jikl} = S_{ijlk} = S_{klij}$$

From the tensor notation of the Hooke's law $\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$, one has

$$
\begin{aligned}
\sigma_{11} &= E_{11kl}\,\varepsilon_{kl} \\
&= E_{1111}\,\varepsilon_{11} + E_{1112}\,\varepsilon_{12} + E_{1113}\,\varepsilon_{13} \\
&\quad + E_{1121}\,\varepsilon_{21} + E_{1122}\,\varepsilon_{22} + E_{1123}\,\varepsilon_{23} \\
&\quad + E_{1131}\,\varepsilon_{31} + E_{1132}\,\varepsilon_{32} + E_{1133}\,\varepsilon_{33} \\
&= E_{1111}\,\varepsilon_{11} + E_{1122}\,\varepsilon_{22} + E_{1133}\,\varepsilon_{33} \\
&\quad + 2E_{1123}\,\varepsilon_{23} + 2E_{1131}\,\varepsilon_{31} + 2E_{1112}\,\varepsilon_{12}
\end{aligned}
$$

$$
= \left[E_{1111},\ E_{1122},\ E_{1133},\ E_{1123},\ E_{1131},\ E_{1112}\right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix}
$$

Similarly, one can calculate all the other stress components.

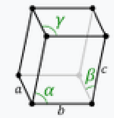
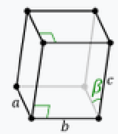
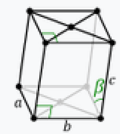
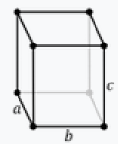
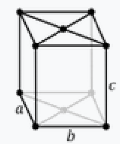
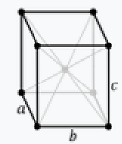
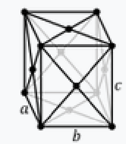
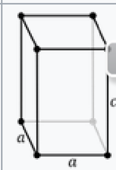
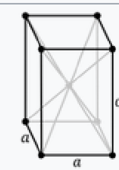
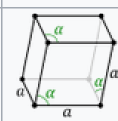
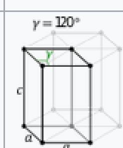
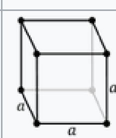
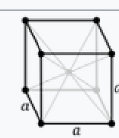
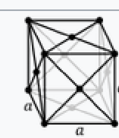
We use the Voigt notation for the stress and strain tensor and can rewrite the Hooke's law in the following matrix notation:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1131} & E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & E_{2223} & E_{2231} & E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & E_{3323} & E_{3331} & E_{3312} \\ E_{2311} & E_{2322} & E_{2333} & E_{2323} & E_{2331} & E_{2312} \\ E_{3111} & E_{3122} & E_{3133} & E_{3123} & E_{3131} & E_{3112} \\ E_{1211} & E_{1222} & E_{1233} & E_{1223} & E_{1231} & E_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix}$$

By $11 \rightarrow 1 \quad 22 \rightarrow 2 \quad 33 \rightarrow 3 \quad 23 \rightarrow 4 \quad 31 \rightarrow 5 \quad 12 \rightarrow 6$,

$$\begin{bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & sym & & E_{44} & E_{45} & E_{46} \\ & & & & E_{55} & E_{56} \\ & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ 2\varepsilon_{4} \\ 2\varepsilon_{5} \\ 2\varepsilon_{6} \end{bmatrix}$$

Thereby, the major symmetry of the stiffness tensor $E_{ijkl} = E_{klij}$ is used.

Crystal family	Lattice system	Schönflies	14 Bravais lattices			
			Primitive (P)	Base-centered (C)	Body-centered (I)	Face-centered (F)
Triclinic		C_i	 aP			
Monoclinic		C_{2h}	 mP	 mS		
Orthorhombic		D_{2h}	 oP	 oS	 oI	 oF
Tetragonal		D_{4h}	 tP		 tI	
Hexagonal	Rhombohedral	D_{3d}	 hR			
	Hexagonal	D_{6h}	 hP			
Cubic		O_h	 cP		 cI	 cF

For materials with certain structure symmetry, the stiffness tensor has additional structure related symmetry, which further decreases the number of independent components. Take the monoclinic materials as an example. They have a symmetric plan, see $x_3 = 0$. Therefore it is expected that

$$E'_{1123} = E_{1123}$$

On the other hand, the stiffness tensor is of the 4th order. In other words, the following transformation tensor relation holds $E\prime_{ijop} = a_{ik}a_{jl}a_{om}a_{pn}E_{klmn}$, where $[a_{ij}]$ is the rotation tensor. Considering the rotation matrix for the coordinate to the one mirrored after its symmetric plan $x_3 = 0$, i.e.

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

one obtains from the transformation relation:

$$E'_{1123} = a_{1m}a_{1n}a_{2p}a_{3q}E_{mnpq} = a_{11}a_{11}a_{22}a_{33}E_{1123} = -E_{1123}$$

Thus, it can only be $E_{1123} = 0$.

Likewise, one can check all the components. It turns out that the number of the independent components of E_{ijkl} for the monoclinic material is reduced to 13, and its matrix notation has the following form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & E_{16} \\ & E_{22} & E_{23} & 0 & 0 & E_{26} \\ & & E_{33} & 0 & 0 & E_{36} \\ & sym & & E_{44} & E_{45} & 0 \\ & & & & E_{55} & 0 \\ & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

Orthotropic materials have three perpendicular symmetric planes i.e. $x_1 = 0, x_2 = 0, x_3 = 0$. They have 9 independent elastic constants, and the stiffness matrix is:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ & E_{22} & E_{23} & 0 & 0 & 0 \\ & & E_{33} & 0 & 0 & 0 \\ & sym & & E_{44} & 0 & 0 \\ & & & & E_{55} & 0 \\ & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

Transversely isotropic materials have three perpendicular symmetric planes $x_1 = 0, x_2 = 0, x_3 = 0$ and additionally one rotational symmetric axis e.g. x_3. They have 5 independent elastic constants, and the stiffness matrix is:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ & E_{11} & E_{13} & 0 & 0 & 0 \\ & & E_{33} & 0 & 0 & 0 \\ & sym & & E_{44} & 0 & 0 \\ & & & & E_{44} & 0 \\ & & & & & \frac{1}{2}(E_{11} - E_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

For cubic materials e.g. Cu, the independent number of components further decreases to 3. The stiffness matrix looks like:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ & E_{11} & E_{12} & 0 & 0 & 0 \\ & & E_{11} & 0 & 0 & 0 \\ & sym & & E_{44} & 0 & 0 \\ & & & & E_{44} & 0 \\ & & & & & E_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

In the case of isotropic materials, there are only 2 independent components/constants.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ & E_{11} & E_{12} & 0 & 0 & 0 \\ & & E_{11} & 0 & 0 & 0 \\ & sym & & \frac{1}{2}(E_{11} - E_{12}) & 0 & 0 \\ & & & & \frac{1}{2}(E_{11} - E_{12}) & 0 \\ & & & & & \frac{1}{2}(E_{11} - E_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

Isotropic stiffness tensor

In fact, any isotropic tensor of the 4th order can be decomposed into two parts and has only two independent constants. Accordingly, the isotropic stiffness tensor can be defined as:

$$\boxed{E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}$$

where λ, μ are the Lamé constants. The stress-strain relation of an isotropic linear elastic material becomes

$\delta_{kl}\varepsilon_{kl} = \varepsilon_{kk}$ $\quad$ $\delta_{ik}\delta_{jl}\varepsilon_{kl} = \varepsilon_{ij}$

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\varepsilon_{kl} = \lambda\varepsilon_{kk}\delta_{ij} + \mu\varepsilon_{ij} + \mu\varepsilon_{ij}$$

$$\boxed{\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}}$$

or

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & sym & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

$$
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ & E_{11} & E_{12} & 0 & 0 & 0 \\ & & E_{11} & 0 & 0 & 0 \\ & sym & & \frac{1}{2}(E_{11} - E_{12}) & 0 & 0 \\ & & & & \frac{1}{2}(E_{11} - E_{12}) & 0 \\ & & & & & \frac{1}{2}(E_{11} - E_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}
$$

$$
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & sym & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}
$$

Comments:

- Comparison with previous representation of the isotropic stiffness matrix leads to

$$E_{11} = \lambda + 2\mu, \qquad E_{12} = \lambda, \qquad E_{44} = \frac{E_{11} - E_{12}}{2} = \mu$$

- By using the decomposition of the strain and the stress,

$$\sigma_{kk} = 3K\varepsilon_{kk}, \quad s_{ij} = 2\mu e_{ij}$$

where K is the bulk modulus

$$K = \lambda + \frac{2}{3}\mu$$

- The principle axes of the stress tensor overlap with those of the strain tensor in the case of isotropic materials.
- Alternatively the material law can be rewritten in

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$

or

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & sym & & \frac{1+\nu}{E} & 0 & 0 \\ & & & & \frac{1+\nu}{E} & 0 \\ & & & & & \frac{1+\nu}{E} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

in which E is the Young's modulus, and ν the Poisson ratio.

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \quad \text{or} \quad \lambda = \frac{\mathrm{E}\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \mathrm{G} = \frac{\mathrm{E}}{2(1+\nu)}$$

Table 1: Relation of elastic constants for isotropic materials (M.E. Gurtin, 1972)

	$E =$	$\nu =$	$\lambda =$	$\mu = G =$	$K =$
E, ν	E	ν	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E}{3(1-2\nu)}$
E, μ	E	$\frac{E}{2\mu} - 1$	$\frac{\mu(E-2\mu)}{(3\mu - E)}$	μ	$\frac{\mu E}{3(3\mu - E)}$
E, K	E	$\frac{1}{2} - \frac{E}{6K}$	$\frac{3K(3K-E)}{(9K-E)}$	$\frac{3KE}{(9K-E)}$	K
ν, λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	ν	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)}{3\nu}$
ν, μ	$2\mu(1+\nu)$	ν	$\frac{2\mu\nu}{1-2\nu}$	μ	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
ν, K	$3K(1-2\nu)$	ν	$\frac{3K\nu}{(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	K
λ, μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	λ	μ	$\frac{3\lambda+2\mu}{3}$
λ, K	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda}{3K-\lambda}$	λ	$\frac{3(K-\lambda)}{2}$	K
μ, K	$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	$\frac{3K-2\mu}{3}$	μ	K

§ 2.6 Strain energy density

We consider the work done dW by a normal stress σ generating a strain $d\varepsilon$ along the stress direction

$$dW = \sigma d\varepsilon$$

The total work accumulated during the quasi-statically deforming from undeformed sate to the current strain state is given as

$$W = \int_0^{\varepsilon} \sigma d\varepsilon = \int_0^{\varepsilon} dU$$

This is the work done throughout the deformation. The last equation is due to the fact, that the elastic energy stored in the material is independent of the deforming path and has to be a total differentiation. It follows,

$$\sigma d\varepsilon = dU = \frac{dU}{d\varepsilon} d\varepsilon \quad \text{sothat} \quad \sigma = \frac{dU}{d\varepsilon}$$

For a 1D linear elastic problem, the strain energy density reads

$$U = W = \int_0^{\varepsilon} \sigma d\varepsilon = \int_0^{\varepsilon} E\varepsilon d\varepsilon = \frac{1}{2} E\varepsilon^2$$

Likewise, one has in general σ_{ij} generating strain $d\varepsilon_{ij}$

$$dW = \sigma_{ij} d\varepsilon_{ij}$$

The accumulated work done is given as a total differentiation:

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} dU$$

Thus,

$$\sigma_{ij} d\varepsilon_{ij} = dU = \frac{\partial U}{\partial \varepsilon_{ij}} d\varepsilon_{ij} \quad \text{sothat} \quad \boxed{\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}}$$

For linear elastic materials, the strain energy density has the form

$$\boxed{U = W = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} = \int_0^{\varepsilon_{ij}} E_{ijkl} \varepsilon_{kl} d\varepsilon_{ij} = \frac{1}{2} E_{ijkl} \varepsilon_{ij} \varepsilon_{kl}}$$

Comment: One replaces the quadratic strain energy density U back into the definition of σ_{ij} and does its differentiation w.r.t. the strain. It follows then

$$\boxed{E_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 U}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = E_{klij}}$$

It explains then the major symmetry of the symmetry of the stiffness tensor.

Notice that $U \geq 0$. $U = 0$ only when $\varepsilon_{ij} = 0$, i.e. any elastic deformation must increase elastic strain energy. This puts some constraints on elastic constants. For isotropic materials, we must have Young's modulus $E > 0$, Possion's ratio $-1 < \nu < 0.5$, and shear modulus $\mu > 0$.

§ 2.7 Summary of the linear elasticity theory

a) Kinematic relation

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

b) Linear momentum balance

$$\sigma_{ij,j} + f_i = 0$$

c) Hooke's law

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$$

or in the isotropic case

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

d) Compatibility condition

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

e) Boundary conditions

$$\sigma_{ij}n_j = t_i^* \quad \text{on } A_t$$

$$u_i = u_i^* \quad \text{on } A_u$$

or mixed type.

Static Lamé-Navier equation

By replacing the kinematic relation into the Hooke's law and then into the balance equation:

$$
\begin{aligned}
0 &= \sigma_{ij,j} + f_i \\
&= \left(2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}\right)_{,j} + f_i \\
&= \left(\mu\left[u_{i,j} + u_{j,i}\right] + \lambda\varepsilon_{kk}\delta_{ij}\right)_{,j} + f_i \\
&= \mu\left[u_{i,jj} + u_{j,ij}\right] + \lambda u_{k,kj}\delta_{ij} + f_i \\
&= \mu\left[u_{i,jj} + u_{j,ij}\right] + \lambda u_{k,ki} + f_i \\
&= \mu\left[u_{i,jj} + u_{j,ji}\right] + \lambda u_{j,ji} + f_i
\end{aligned}
$$

It leads to the 2nd order partial differntial equations w.r.t. the displacements (3 component equations for 3 displacement components):

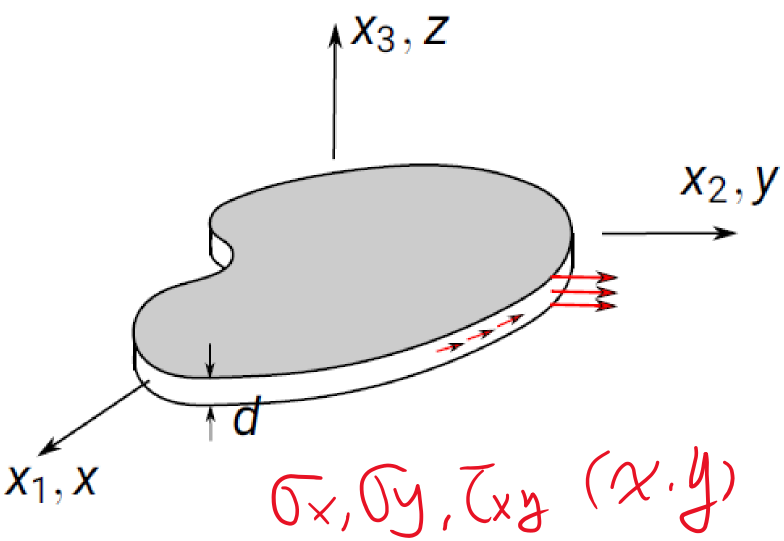
$$0 = (\mu + \lambda)u_{j,ji} + \mu u_{i,jj} + f_i$$

On the other hand, one can express the strain by stress and replace the results into the compatibility equation:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}f_{k,k}\delta_{ij} - (f_{j,i} + f_{i,j})$$

§ 2.8 2D problems

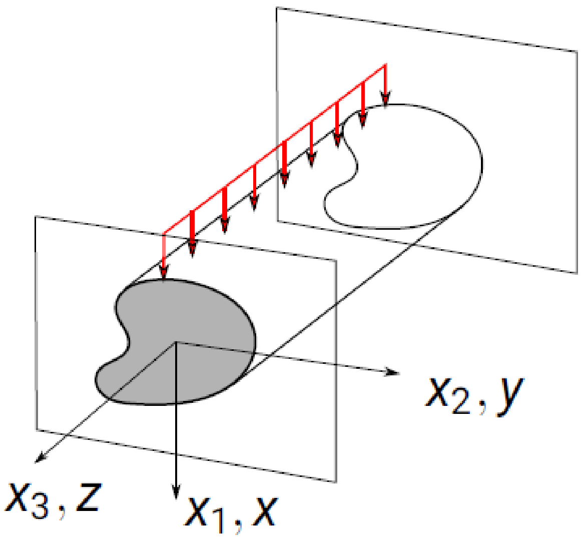
Plane stress problem:



- A thin layer with thickness d, which is much smaller in comparison with the length in the plane.
- only the sides are prescribed with boundary conditions parallel to the plane; the top and the bottom sides are free.
- Assume $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$.

Only left are $\sigma_x, \sigma_y, \tau_{xy}$, and they are only functions of the in-plane coordinates x, y. The displacements u, v are thus also only dependent on x, y. It should be noted that the displacement w in the z direction does not vanish, as well as the out-plane strain ε_z.

Plane strain problem:



- The cross section and the loading remain uniform along the *z* direction.
- Only the side surfaces are subjected to loading, which lies parallel to the cross section.
- Displacement *w* in *z* direction is zero, and other displacement components are also independent of *z*.

Thus

$$u = u(x, y), v = v(x, y), w = 0$$

$$\varepsilon_z = \varepsilon_{zx} = \varepsilon_{zy} = 0$$

The strains $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ and the stresses $\sigma_x, \sigma_y, \tau_{xy}$ are only functions of *x* and *y*. It is worth to mention that the normal stress σ_z does not vanish.

2D linear isotropic elasticity

a) Kinematic relation

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)$$

b) Linear momentum balance

$$\sigma_{\alpha\beta,\beta} + f_\alpha = 0$$

c) linear isotropic material law

$$\varepsilon_{\alpha\beta} = -\frac{\nu}{E}\sigma\gamma\gamma\delta_{\alpha\beta} + \frac{1+\nu}{E}\sigma_{\alpha\beta}$$

d) Compatibility condition

$$\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0$$

e) Boundary conditions

$$\sigma_{\alpha\beta}n_\beta = t^*_\alpha \quad \text{on } A_t$$

$$u_\alpha = u^*_\alpha \quad \text{on } A_u$$

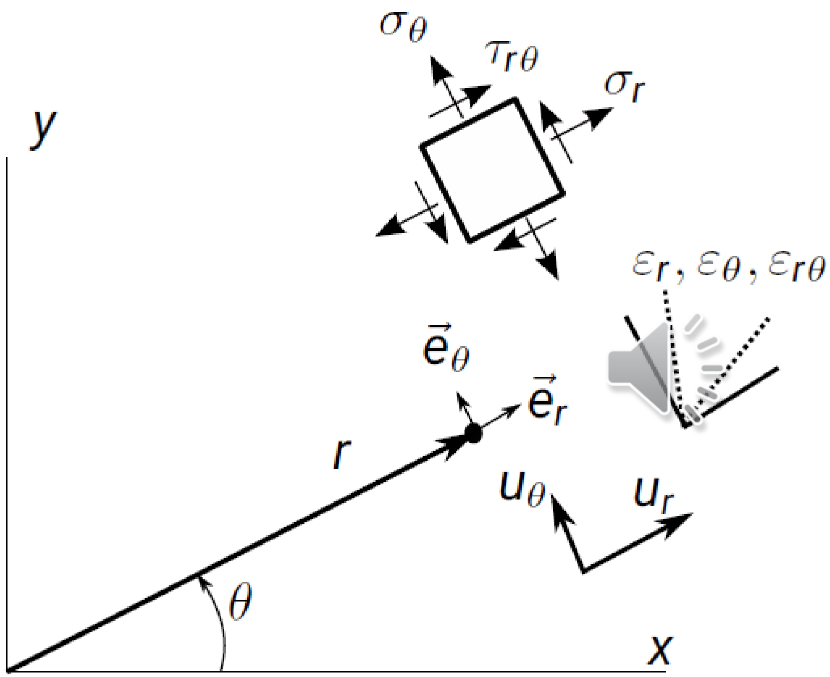
or mixed type.

Hereby $\alpha, \beta, \gamma = 1, 2$, and

$$E \to E,\ \nu \to \nu \quad \text{for plane stress problem}$$

$$E \to \frac{E}{1-\nu^2},\ \nu \to \frac{\nu}{1-\nu} \quad \text{for plane strain problem}$$

2D linear isotropic elasticity in the polar coordinate system



$$(x, y) \leftrightarrow (r, \theta)$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

By using the identities $x = r\cos\theta, \quad y = r\sin\theta$, and the resultant relations between the derivatives $\partial x, \partial y$ and $\partial r, \partial\theta$, one can rewrite the governing equations w.r.t the polar coordinates:

linear momentum balance:

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0$$

kinematics:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad \gamma_{r\theta} = 2\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)$$

linear elastic isotropic material law:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r), \quad \varepsilon_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta}$$

Replacing the isotropic Hooke's law into the compatibility equation,

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

one has

$$\frac{1}{E}\left(\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right) = \frac{2(1+\nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$E \to E/(1-\nu^2)$

$\nu \to \nu/(1-\nu)$

This can be further simplified. In fact, from the balance equation without body force,

$$\sigma_{x,x} + \tau_{xy,y} = 0 \quad \sigma_{y,y} + \tau_{xy,x} = 0$$

one can take derivative of the first equation w.r.t x, and the second w.r.t. y:

$$\sigma_{x,xx} + \tau_{xy,yx} = 0, \quad \sigma_{y,yy} + \tau_{xy,xy} = 0$$

Addition of these two eqs. leads to

$$\sigma_{x,xx} + \sigma_{y,yy} = -2\tau_{xy,yx} \quad \text{or} \quad \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = -2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

Replace this result into the compatibility condition expressed by stress components.

One has

$$\left(\frac{\partial^2\sigma_x}{\partial y^2} - \nu\frac{\partial^2\sigma_y}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} - \nu\frac{\partial^2\sigma_x}{\partial x^2}\right) = -(1+\nu)\left(\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2}\right)$$

After simplification,

$$\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial x^2} + \frac{\partial^2\sigma_x}{\partial y^2} + \frac{\partial^2\sigma_y}{\partial x^2} = 0$$

By using the Lapace-Operator $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, the last equation is rewritten into

$$\boxed{\Delta\left(\sigma_x + \sigma_y\right) = 0}$$

$\leftarrow \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$

Combination of this equation and the 2 stress equilibrium equations leads to a 3-equation system for the three unknown stress components $\sigma_x, \sigma_y, \tau_{xy}$.

Airy stress function

The number of equations can be further reduced by introducing the so-called Airy stress function $F = F(x, y)$ according to George Biddel Airy:

$$\boxed{\sigma_x = \frac{\partial^2 F}{\partial y^2}, \sigma_y = \frac{\partial^2 F}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 F}{\partial y \partial x}}$$

If the stress components can be fined from one Airy function $F(x, y)$ in this way, the two stress equilibrium equations are automatically fulfilled ($f_i = 0$):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \frac{\partial^3 F}{\partial x \partial y^2} - \frac{\partial^3 F}{\partial^2 y \partial x} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = \frac{\partial^3 F}{\partial y \partial x^2} - \frac{\partial^3 F}{\partial^2 x \partial y} = 0$$

It can be shown that $\sigma_x + \sigma_y = \Delta F$. Inserting this into the compatibility condition, one has

$$\boxed{\Delta\left(\sigma_x + \sigma_y\right) = \Delta(\Delta F) = 0}$$

$$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)F = 0$$

$$\frac{\partial^4 F}{\partial x^4} + \frac{\partial^4 F}{\partial y^4} + \frac{\partial^4 F}{\partial x^2 \partial y^2} = 0$$

This is a biharmonic equation and is a 4th order partial differential equation.

In the polar coordinates, the Lapalace operator is

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial^2 \theta}$$

Thus the biharmonic equation becomes:

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r})(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r})F(r,\theta) = 0$$

The stress components are given in

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r}\frac{\partial F}{\partial r}, \quad \sigma_\theta = \frac{\partial^2 F}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)$$

§ 2.9 Illustrative examples
Isotropic circular inhomog. in isotropic matrix

TECHNISCHE UNIVERSITÄT DARMSTADT

For rotational symmetric plane stress problem, the equations are reduced to:

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\varphi) = 0$$

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\varphi = \frac{u_r}{r}$$

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\varphi),$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}(\varepsilon_\varphi + \nu\varepsilon_r),$$

$$\tau_{r\varphi} = 0$$

$u_r(r)$
$\varepsilon_r(r), \varepsilon_\varphi(r)$
$\sigma_r(r), \sigma_\varphi(r)$

linear momentum balance:

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0$$

$f_i = 0$

kinematics:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad \gamma_{r\theta} = 2\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)$$

linear elastic isotropic material law:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r), \quad \varepsilon_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta}$$

$$\rightarrow \quad \frac{\mathrm{d}^2u_r}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u_r}{\mathrm{d}r} - \frac{u_r}{r^2} = 0$$

$$u_r = C_1 r + \frac{C_2}{r}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2}\right]$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2}\right]$$

C_1, C_2 are the integration constants.

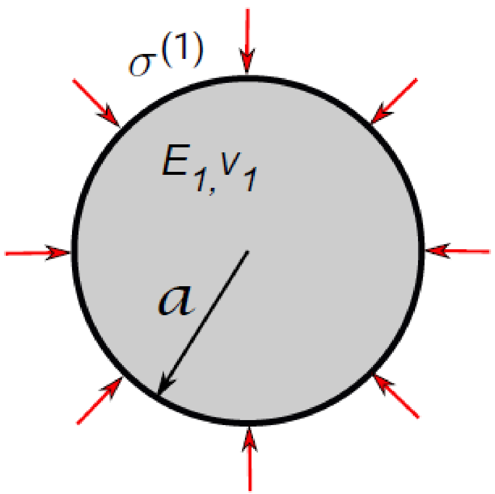
MECHANICS of FUNCTIONAL MATERIALS

Problem 1:

Boundary conditions:

$$\sigma_r(0) = limited. \quad \rightarrow \quad C_2 = 0$$

$$\sigma_r(a) = \sigma^{(1)} \quad \rightarrow \quad C_1 = \frac{\sigma^{(1)}}{E_1}(1-\nu_1)$$



$$\rightarrow \quad \sigma_r = \sigma_\varphi = \sigma^{(1)} = const.\ !$$

$$u_r = C_1 r = \frac{\sigma^{(1)}}{E_1}(1-\nu_1)r, \quad u_r(a) = \frac{\sigma^{(1)}a}{E_1}(1-\nu_1)$$

$$\varepsilon_r = \varepsilon_\varphi = C_1 = \frac{\sigma^{(1)}}{E_1}(1-\nu_1)$$

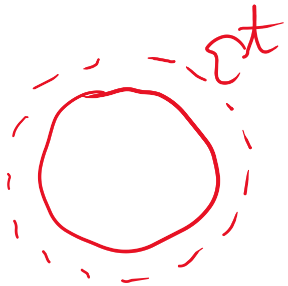
$$u_r = C_1 r + \frac{C_2}{r}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2}\right]$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2}\right]$$

The stress and strain fields are uniform. Every point experiences pure hydrostatic stress state and pure volumetric deformation.

Problem 1a: If there is additional inelastic isotropic strain ε^t due to e.g. temperature or phase transformation:



$$\sigma_r = \sigma_\varphi = 0$$

$$\varepsilon_r = \varepsilon_\varphi = \varepsilon^t$$

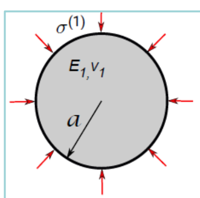
$$u_r = \varepsilon^t r$$

Problem 1b: $\sigma^{(1)}$ and ε^t appear at the same time

$$\sigma_r = \sigma_\varphi = \sigma^{(1)} = const.$$

$$\varepsilon_r = \varepsilon_\varphi = \varepsilon^t + C_1 = \varepsilon^t + \frac{\sigma^{(1)}}{E_1}(1 - \nu_1)$$

$$u_r = (C_1 + \varepsilon^t)r = [\frac{\sigma^{(1)}}{E_1}(1 - \nu_1) + \varepsilon^t]r$$



$\rightarrow \sigma_r = \sigma_\varphi = \sigma^{(1)} = const.\ !$

$$u_r = C_1 r = \frac{\sigma^{(1)}}{E_1}(1 - \nu_1)r,$$

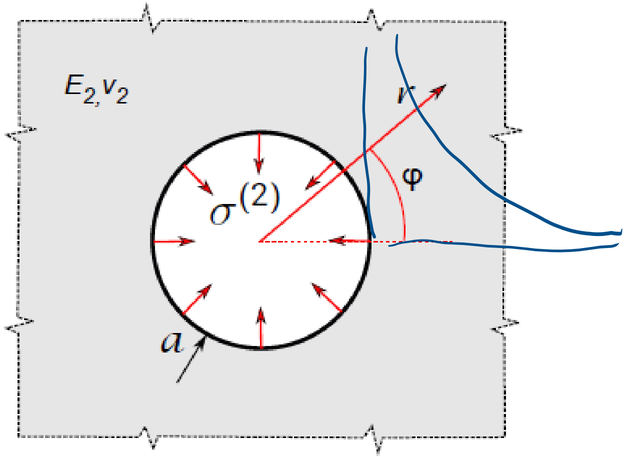
$$\varepsilon_r = \varepsilon_\varphi = C_1 = \frac{\sigma^{(1)}}{E_1}(1 - \nu_1)$$

$$u_r = C_1 r + \frac{C_2}{r}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2}\right]$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left[(1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2}\right]$$

Problem 2:



Boundary condition:

$$\sigma_r(\infty) \to 0 \quad \to \quad C_1 = 0$$

$$\sigma_r(a) = \sigma^{(2)} \quad \to \quad C_2 = -\frac{\sigma^{(2)}}{E_2}(1+\nu_2)a^2$$

$$\sigma_r = -\sigma_\varphi = \sigma^{(2)}\frac{a^2}{r^2}, \qquad u_r = \frac{C_2}{r} = -\frac{\sigma^{(2)}}{E_2}(1+\nu_2)\frac{a^2}{r},$$

$$\varepsilon_r = -\varepsilon_\varphi = -\frac{C_2}{r^2} = \frac{\sigma^{(2)}}{E_2}(1+\nu_2)\frac{a^2}{r^2}$$

Comments: The stress loading $\sigma^{(2)}$ is a kind of self-equilibrium loading, which fulfills alone the equilibrium. For this type of equilibrium loading,

$$\sigma_r \propto 1/r^2, \quad u_r \propto 1/r, \quad \varepsilon \propto 1/r^2 \quad \text{in 2D}$$

$$\sigma_r \propto 1/r^3, \quad u_r \propto 1/r^2, \quad \varepsilon \propto 1/r^3 \quad \text{in 3D}$$

$$\sigma_r = -\sigma_\varphi = \sigma^{(2)}\frac{a^2}{r^2}, \qquad u_r = \frac{C_2}{r} = -\frac{\sigma^{(2)}}{E_2}(1+\nu_2)\frac{a^2}{r},$$

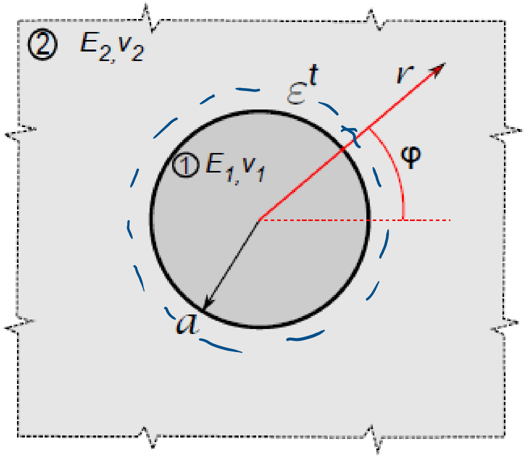
$$\varepsilon_r = -\varepsilon_\varphi = -\frac{C_2}{r^2} = \frac{\sigma^{(2)}}{E_2}(1+\nu_2)\frac{a^2}{r^2}$$

$$\sigma_r = \sigma_\varphi = \sigma^{(1)} = const.$$

$$\varepsilon_r = \varepsilon_\varphi = \varepsilon^t + C_1 = \varepsilon^t + \frac{\sigma^{(1)}}{E_1}(1-\nu_1)$$

$$u_r = (C_1+\varepsilon^t)r = [\frac{\sigma^{(1)}}{E_1}(1-\nu_1)+\varepsilon^t]r$$

Problem 3: Only loaded by isotropic transformation strain ε^t



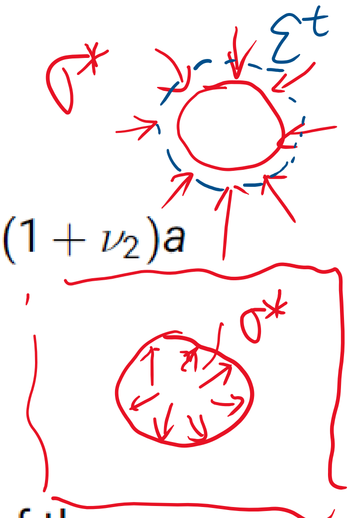
Transition conditions:

$$u_r^{(1)}(a) = u_r^{(2)}(a)$$

$$\sigma^{(1)} = \sigma^{(2)} = \sigma^*$$

$$\rightarrow \quad [\frac{\sigma^*}{E_1}(1-\nu_1)+\varepsilon^t]a = -\frac{\sigma^*}{E_2}(1+\nu_2)a$$

$$\rightarrow \quad \sigma^* = -\frac{\varepsilon^t}{\frac{(1-\nu_1)}{E_1}+\frac{(1+\nu_2)}{E_2}}$$

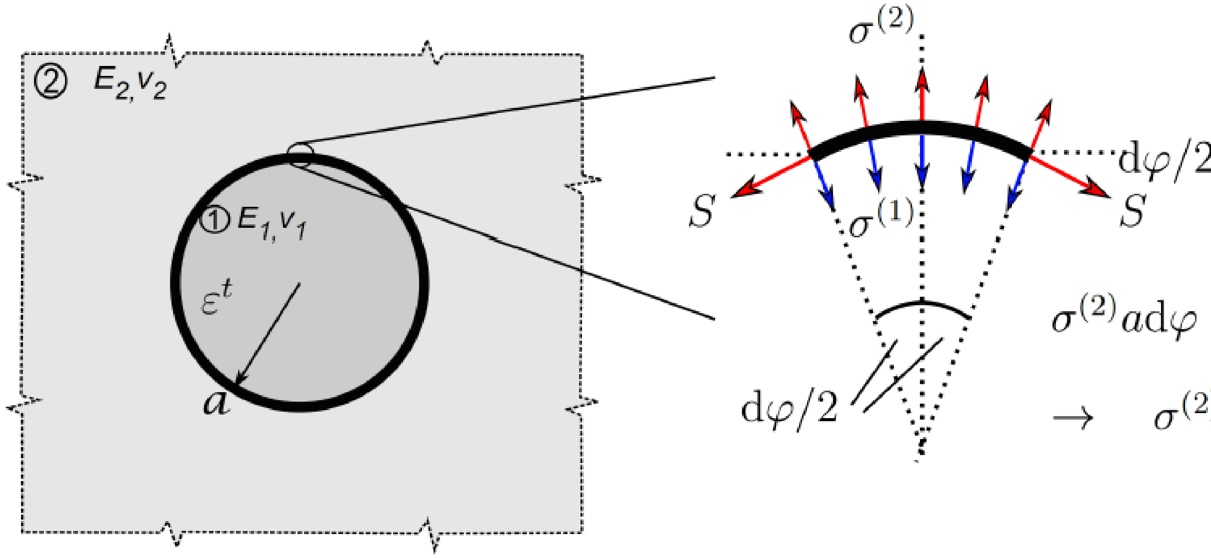


Insertion of $\sigma^* = \sigma^{(1)} = \sigma^{(2)}$ into the solutions of the Problem 1b and of the Problems 2 leads to the solution of the interior and exterior region. For instance, the strain in the inhomogeneity is

$$\varepsilon_r^{(1)} = \varepsilon_\varphi^{(1)} = \varepsilon^t\left[1-\frac{1}{E_1}(1-\nu_1)\frac{1}{\frac{(1-\nu_1)}{E_1}+\frac{(1+\nu_2)}{E_2}}\right]$$

$$E_1 = E_2 = E, \nu_1 = \nu_2 = \nu \quad \rightarrow \quad \sigma^* = -\frac{E}{2}\varepsilon^t, \;\; \varepsilon_r^{(1)} = \varepsilon_\varphi^{(1)} = \frac{1+\nu}{2}\varepsilon^t$$

Problem 4: with interface stress S between the region 1 and the region 2



$$\sigma^{(2)} a \mathrm{d}\varphi - \sigma^{(1)} a \mathrm{d}\varphi - S \mathrm{d}\varphi = 0$$

$$\rightarrow \quad \sigma^{(2)} - \sigma^{(1)} = \frac{S}{a}$$

Transition conditions:

$$\sigma^{(2)} - \sigma^{(1)} = \frac{S}{a}, \quad u_r^{(1)}(a) = u_r^{(2)}(a)$$

$$\rightarrow \quad \sigma^{(1)} \left[\frac{1-\nu_1}{E_1} + \frac{1+\nu_2}{E_2} \right] = -\varepsilon^t - \frac{S}{a} \frac{1+\nu_2}{E_2}$$

Particularly for: $E_1 = E_2 = E, \nu_1 = \nu_2 = \nu$

$$\sigma^{(1)} = -\frac{1}{2} E \varepsilon^t - \frac{1}{2} \frac{S}{a} (1+\nu)$$