Grammarforms and Grammarhomomorphisms

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Summary. We show that grammar forms, introduced by S. Ginsburg, can be treated with the help of grammar homomorphisms, introduced by G. Hotz. This gives us the possibility to generalize the notion of grammar forms in such a way that we can prove closure properties of the associated language family using some simple axioms by diagram chasing. Moreover we demonstrate a principle of duality in dealing with this theory.

Introduction

There has been an extensive study of families of languages and automata. In contrast to this fact, there is no appropriate theory of grammar families. Recently, Ginsburg et al. [3], introduced a theory of grammar families using grammarforms. By this approach grammar families are generated via interpretations of a given grammar, called grammarform, which reflects the structural properties of the productions of the interpreted grammar.

The aim of this paper is to demonstrate how this approach can be embedded in the theory of grammar homomorphisms, introduced by G. Hotz [5, 6, 8].

We'll show that the mechanism of grammar forms can be generalized in a natural way such that closure properties of the generated language family can be derived by diagram chasing. Moreover this generalization demonstrates a principle of duality.

1. Grammars and X-Categories

We give some basic definitions and a short review of the construction of X-categories ([5, 8]).

A semi-Thue-system is a pair $S=(A(S), P(S))$ such that $A(S)$ is an alphabet and $P(S)$ is a finite subset of $A(S)^* \times A(S)^*$ where $A(S)^*$ is the free monoid over $A(S)$.

Free X-categories are precise definitions of "derivations" associated with semi-Thue-systems.

Let $w, v \in A(S)^*$: $w$ is derivable into $v (w \vdash v)$ (with respect to $S$), if there exists a sequence.

$$w = u_0 q_0 q_0, \quad u_0 q_0 v_0 = u_1 p_1 v_1, \ldots, \quad u_r q_{r-1} v_{r-1} = u_r p_r v_r, \quad u_r q_r v_r = v$$

where $r \geq 0$, $u_i$, $v_i$, $p_i$, $q_i \in A(S)^*$ ($0 \leq i \leq r$) and $(p_i, q_i) \in P(S)$ ($0 \leq i \leq r$).
This sequence is called a derivation. The set of all derivations defines a free 
X-category $F(S)$. Informally $F(S)$ can be described as follows:
the objects are the elements of $A(S)^*$,
the morphisms are the “derivations”,
if $f = (w \mapsto v)$ is a derivation, then
$\overline{d}f = w$ is the domain of $f$ and
$\overline{c}f = v$ is the codomain,
the empty derivations ($r = 0$) are the identities $\mathbb{1}_w$,
the operation “$\circ$” is the concatenation of derivations illustrated by the
\[
(w \mapsto v \mapsto z = g \circ f)
\]
the monoidal operation “$\times$” is illustrated by the diagram
\[
\begin{bmatrix}
\overline{w} & \overline{v} \\
\overline{w_1} & \overline{v_1}
\end{bmatrix} = f \times g.
\]
Note, that in a natural way $P(S)$ can be embedded into the set of morphisms
of $F(S)$, such that $d(p, q) = p$ and $c(p, q) = q$.

A functor between $X$-categories is a covariant functor, which is a monoid-
homomorphism with respect to the monoidal operation. A functor between $F(S)$
and another $X$-category $\mathcal{X}$ is now uniquely determined by exhibiting a monoid-
homomorphism $\phi_0$ between the objects and a mapping $\phi_M$ from $P(S)$ into the
morphism-set of $\mathcal{X}$, such that
\[
c, d(\phi'_M(r)) = \phi_0(c, d(r))
\]
for all $r \in P(S)$.

We adopt some further notations.

If $f$ is a morphism of $F(S)$, we write $f \in F(S)$. Now, consider two subsets $L_1$
and $L_2$ of $A(S)^*$, then
\[
(L_1, L_2)_S = \{f \in F(S) | \overline{d}f \in L_1 \& \overline{c}f \in L_2\}.
\]
If $f \in F(S)$ then $f$ can be represented in the form
\[
f = (1_{w_1} \times r_1 \times 1_{v_1}) \cdots (1_{w_s} \times r_s \times 1_{v_s})
\]
\[
(s \geq 0, w_i, v_i \in A(S)^*, r_i \in P(S), 1 \leq i \leq s).
\]
This representation is called a sequential representation of $f$. The number $s$
is independent a special representation of $f$. It is called the length $\|f\|$ of $f$.

Obviously,

i) $\|f\| = 0 \iff f$ is an identity,

ii) $\|f \circ g\| = \|f\| + \|g\|$.

iii) $\|f \times g\| = \|f\| + \|g\|$.

For further details the reader is referred to [8]. A grammar $G$ is a 4-tuple
\[
G = (A(G), P(G), T(G), s(G))
\]
where

i) \((A(G), P(G))\) is a semi-Thue-system,

ii) \(P(G) \subseteq Z(G)^* \times A(G)^*\) where
\[
Z(G) = A(G) \setminus T(G) \quad \text{and} \quad Z(G)^* = Z(G)^* \setminus \{\varepsilon\},
\]

iii) \(T(G) \cup s(G) \subseteq A(G) \& s(G) \subseteq Z(G)\).

As usual, \(A(G)\) is the alphabet, \(T(G)\) the set of terminals, \(Z(G)\) the set of variables, \(s(G)\) the set of axioms and \(P(G)\) the set of productions of \(G\). With respect to \(X\)-categories we transfer our notation from semi-Thue-systems to grammars. Then

\[
D(G) = (s(G), T(G)^*)_G
\]

is the set of derivations of \(G\), and

\[
\mathcal{L}(G) = e(D(G))
\]

is the generated language.

We assume that the reader is familiar with standard notions of the language theory [2].

2. Grammar Homomorphisms

We discuss algebraic concepts related to the use of \(X\)-categories.

A more extensive study can be found in [1] and [11]. Consider two grammars \(G_1\) and \(G_2\).

A homomorphism \(\phi: G_1 \rightarrow G_2\) is a functor \(\phi: F(G_1) \rightarrow F(G_2)\) with

i) \(\phi(T(G_1)) \subseteq T(G_2)^*\),

ii) \(\phi(s(G_1)) \subseteq s(G_2)\),

iii) \(\phi(Z(G_1)) \subseteq Z(G_2)\).

Remark. If \(\phi: G_1 \rightarrow G_2\) is a homomorphism, then \(\phi(D(G_1)) \subseteq D(G_2)\) and therefore \(\phi(\mathcal{L}(G_1)) \subseteq \mathcal{L}(G_2)\).

We now briefly discuss various constructions, which can serve as examples for special classes of homomorphisms.

Consider triples \((Z, T, s)\) of alphabets such that \(Z \cap T = \phi\) and \(s \subseteq Z\).

A monoidhomomorphism

\[
h: (Z_1 \cup T_1)^* \rightarrow (Z_2 \cup T_2)^*
\]

is called admissible if

i) \(h(T_1) \subseteq T_2^*\),

ii) \(h(s_1) \subseteq s_2\) and

iii) \(h(Z_1) \subseteq Z_2\).

If \(G\) is a grammar,

\[
h: (Z(G), T(G), s(G)) \rightarrow (Z, T, s)
\]
is admissible, then a grammar $G_1$ and a homomorphism $\phi: G \to G_1$ are defined by

i) $Z(G_1) = Z, \quad s(G_1) = s, \quad T(G_1) = T,$

ii) $P(G_1) = \{ (h(p), h(q)) \mid (p, q) \in P(G) \},$

iii) $\phi$ is on objects equal to $h$, and $\phi(p, q) = (h(p), h(q)) \quad ((p, q) \in P(G)).$

**Fact.** $\phi(P(G)) = P(G_1).$

This observation leads to the following definition.

**Definition 2.1.** $\phi: G_1 \to G_2$ a homomorphism.

i) $\phi$ is a *fine* homomorphism iff $\phi(P(G_1)) \subseteq P(G_2),$

ii) $\phi$ is a *very fine* homomorphism iff $\phi(T(G_1)) \subseteq T(G_2)$ & $\phi$ is fine.

We can prove more.

**Lemma 2.1.** If $h$ is an admissible $\varepsilon$-free monoidhomomorphism then $\phi_h(D(G)) = D(G_1)$ and therefore $h(L(G)) = L(G_1)$, if $h$ is injective on variables and $h^{-1}(s(G_1)) = s(G).$

**Proof.** We consider the set

$$\hat{D}(G) = \{ f \in F(G) \mid \| f \| \geqslant 1, \; d \| f \| \rangle s(G) \; c \in A(G)^* \}$$

and the corresponding set $\hat{D}(G_1).$

If we can show that $\phi_h(\hat{D}(G)) = \hat{D}(G_1)$ we get the desired result.

We therefore show by induction on $\| f \|$, that to any $f' \in \hat{D}(G_1)$ there exists $f \in \hat{D}(G)$ with $\phi_h(f) = f'$. "\| f' \| = 1":

In this case

$$f' = (\sigma, q) \in P(G_1) \quad \text{with} \; \sigma \in s(G_1).$$

Since $\phi_h(P(G)) = P(G_1)$ we find $(\sigma', q') \in P(G)$ with $\phi_h((\sigma', q')) = (\sigma, q).$

Moreover $\sigma' \in h^{-1}(\sigma) \subseteq h^{-1}(s(G_1)) = s(G).

"\| f' \| = k \Rightarrow \| f \| = k + 1":$

We consider a decomposition $f' = (1_{u'} \times r' \times 1_v) \circ f_1$ with $r' \in P(G)$ and $f_1 \in \hat{D}(G_1).

By the induction hypothesis there exists $f_1 \in \hat{D}(G)$ with

$$f_1 = \phi_h(f_1), \quad c(f_1) = u' \cdot v \quad \text{and} \quad h(u') = u', \quad h(v) = v'.$$

On the other hand there exists a production $r = (p', q')$ with $\phi_h(r) = r'$.

Now we know

$$h(p) = d(r') = h(p').$$

Since $h$ is injective on variables we get $p = p'$; but then $\phi((1_u \times r \times 1_v) \circ f_1) = f'$, which proves our lemma.

**Definition 2.2.** Let be $\phi: G_1 \to G_2$ a homomorphism.

i) $\phi$ is *epic* iff $\phi(P(G_1)) = P(G_2),$

ii) $\phi$ is *closed* iff $\phi(D(G_1)) = D(G_2).

A modification of the proof of Lemma 2.1 yields
Lemma 2.2. If $\phi: G_1 \rightarrow G_2$ is a fine homomorphism, with $\phi$ injective on variables, then $\phi$ is closed, if $\phi^{-1}(s(G_2)) = s(G_1)$.

Consider in contrast to the above construction the following situation. Let $G$ a grammar and

$$h: (Z, T, s) \rightarrow (Z(G), T(G), s(G))$$

admissible. We assume, $h$ to be $e$-free, then we can define a grammar $G^h$ by

1) $Z(G^h) = Z$, $T(G^h) = T$, $s(G^h) = s$.
2) $P(G^h) = \{(h(q), h(r)) | (q, r) \in P(G)\}$.

Note, that in this case $(G^h)_h = G$; therefore, there is a fine homomorphism $\phi^h: G^h \rightarrow G$.

The homomorphisms $\phi^h$ are special cases of another type of closed homomorphisms, introduced by G. Hotz [7], [6].

Definition 2.3. Let be $\phi: G_1 \rightarrow G_2$ a homomorphism.

i) $\phi$ is normal if and only if

$$\langle p, q \rangle \in P(G_1), \quad \phi^p \in A(G_1)^*, \quad \phi(p) = \phi^p \text{ implies } \langle p', q \rangle \in P(G_1).$$

ii) $\phi$ is conormal if and only if

$$\langle p, q \rangle \in P(G_1), \quad q^r \in A(G_1)^*, \quad \phi(q) = \phi(q^r) \text{ implies } \langle p, q^r \rangle \in P(G_1).$$

Fact. $\phi_h$ is both normal and conormal and epic.

Lemma 2.3. If $\phi: G_1 \rightarrow G_2$ is epic and normal, then $\phi$ is closed, if

$$\phi^{-1}(s(G_2)) = s(G_1).$$

Proof. Without loss of generality we can assume $\phi$ to be epic. We show by induction on the length of derivations, that to any $f' \in \hat{D}(G_2)$ there exists $f \in \hat{D}(G_1)$ with $\phi(f) = f'$.

If $\|f'\| = 1$, there is nothing to prove.

Now assume

$$f' = (t_1 \times r' \times 1_1) \circ f_1'.$$

Then there exists a derivation $f_1 \in \hat{D}(G_1)$ with $\phi(f_1) = f_1'$. Since $\phi$ is epic there exists $r \in P(G_2)$ with $\phi(r) = r'$. We know

$$c_{f_1} = uv$$

with

$$\phi(u) = u', \quad \phi(v) = v', \quad \phi(\phi) = \phi(d(r) = d(r').$$

Since $\phi$ is normal, we get $\langle \phi, c(r) \rangle \in P(G_1)$.

Moreover

$$\phi((p, c(r))) = (\phi(p), \phi(c(r))) = \phi(r) = r'.$$

By this we get

$$\phi((1_1 \times (p, c(r))(1_1) \circ f_1) = f'.$$

In the case of conormal homomorphisms we can show more
Lemma 2.4. If $\phi : G_1 \rightarrow G_2$ is conormal and epic, then $\phi$ is closed and $\phi(L(G_1)) = L(G_2)$ if $\phi^{-1}(s(G_2)) = s(G_1)$.

Proof. We show:

To any $w \in A(G_1)^*$ such that there exists a derivation $f' \in D(G_2)$ with $c(f') = \phi(w)$, there exists a derivation $f \in D(G_1)$ with $\phi(f) = f'$ and $c(f) = w$. Again we apply induction on the length of $f'$.

If $|f'| = 1$, then $(\sigma', \phi(w)) \in P(G_2)$ with $\sigma' \in s(G_2)$. But then there exists $(\sigma, w_1) \in P(G_1)$ with $\phi(\sigma) = \sigma'$ and $\phi(w_1) = \phi(w)$. Since $\phi$ is conormal $(\sigma, w) \in P(G_1)$.

If $|f'| > 1$, we consider a decomposition

$$f' = (1_{u', q'} \times 1_{v'}) \circ f_1'.$$

We know $w = uqv$ with $\phi(u) = u'$, $\phi(q) = q'$ and $\phi(v) = v'$. Then there is a production $(\phi, q')$ with $\phi(\phi) = \phi'$ and $\phi(q) = q'$.

Since $\phi$ is conormal we get $r = (\phi, q') \in P(G_1)$ and $\phi(r) = (\phi', q')$.

Now consider the word $u\phi v$; then $\phi(u\phi v) = d(f'_1)$. By the induction hypothesis there is a derivation $f_1 \in D(G_1)$ with $\phi(f_1) = f'_1$ and $c(f_1) = u\phi v$. But then we can build

$$f = (1_u \times (\phi, q) \times 1_v) \circ f_1$$

which gives the desired result.

Examples of very fine homomorphisms $\phi$ with

i) $\phi$ is surjective on objects,

ii) $\phi$ is epic,

iii) $\phi$ is not a closed homomorphism

are given by the classical intersection theorem.

Theorem 2.1. If $G$ is a grammar, $R \subseteq T(G)^*$ a regular set, then there exist a grammar $G_R$ and a very fine homomorphism $\phi_R : G_R \rightarrow G$ with

i) $\phi_R(P(G_R)) = P(G),$

ii) $\phi_R$ is surjective on objects,

iii) $\phi_R$ is the identity on terminals,

iv) $L(G_R) = L(G) \cap R.$

Proof. We use a slightly modified version of the classical construction of $G_R$ [2]. Consider a finite state acceptor $a = (Q, X, \delta, F, q_0)$ with $T(a) = R$.

Now define $G_R$ by

1) $Z(G_R) = Q \times Z(G) \times Q,$

$$T(G_R) = T(G),$$

$$s(G_R) = (q_0) \times s(G) \times F.$$

2) $P(G_R)$ is constituted by the following productions. Let $\eta_1 \ldots \eta_s \rightarrow t_0 \xi_1 \ldots \xi_r \in P(G)$ with

$$s > 0, \quad r \geq 0, \quad \eta_i \in Z(G) \ (1 \leq i \leq s), \quad \xi_j \in Z(G),$$

$$(1 \leq j \leq r), \quad t_j \in T(G)^* \ (0 \leq j \leq r).$$
Then $$\phi_1, \eta_1, \phi'_1 \ldots, \phi_r, \eta_r, \phi'_r \rightarrow t_0(q_1, \xi_1, q'_1), t_1 \ldots(q_r, \xi_r, q'_r) t_r \in P(G_1)$$
where $$r \geq 0$$, $$\phi_i, \phi'_i, q_i, q'_i \in Q$$ and

$$q_1 = \delta(t_0, \phi_1) \land q_{i+1} = \delta(t_i, q'_i) \quad (1 \leq i \leq r) \land \phi'_r = \delta(t_r, q'_r)$$

if $$r = 0$$ and

$$\phi'_r = \delta(t_0, \phi_1)$$

if $$r = 0$$.

The homomorphism $$\phi_R$$ is then given by the admissible monoidhomomorphism

$$h: (Z(G_R), T(G), s(G_R)) \rightarrow (Z(G), T(G), s(G))$$

defined by

$$h((\phi, \xi, q)) = \xi \quad ((\phi, \xi, q) \in Z(G_R))$$

and

$$h(t) = t \quad (t \in T(G_R)).$$

Condition iii) gives rise to another definition. Consider a homomorphism $$\phi: G_1 \rightarrow G_2$$ and a subset $$A \subseteq A(G_1)$$. We call $$\phi$$ an $$A$$-homomorphism iff $$\phi(\xi) = \xi$$, if $$\xi \notin A$$. Of special interest are $$Z$$-morphisms ($$A = Z(G_1)$$) and $$T$$-morphisms ($$A = T(G_1)$$).

We call $$\phi$$ an isomorphism if $$\phi$$ is both on objects and morphisms bijective, ($$G_1 \cong G_2$$ or $$G_1 \equiv G_2$$).

Note that an isomorphism is always very fine and closed.

We are now dealing with "subgrammars". Consider two grammars $$G_1$$ and $$G_2$$.

$$G_1$$ is a subgrammar of $$G(G_1 \subseteq G)$$ if

i) $$Z(G_1) \subseteq Z(G)$$, $$s(G_1) \subseteq s(G)$$, $$T(G_1) \subseteq T(G)$$ and $$P(G_1) \subseteq P(G)$$,

ii) $$Z(G_1) \cap s(G) = s(G_1)$$.

If $$G_1$$ and $$G_2$$ are subgrammars, then the "union" $$G_1 \cup G_2$$ and the "intersection" $$G_1 \cap G_2$$ are defined in a natural way. More difficult to deal with is the concept of image and coimage.

If $$\phi: G_1 \rightarrow G_2$$ is a homomorphism, $$G'_1 \subseteq G_1$$ and $$G'_2 \subseteq G_2$$ then

$$\phi(G'_1) = \bigcap \{ \tilde{G}_2 \subseteq G_2 \mid \phi((A(G_1)^*, A(\tilde{G}_1)^*) G_1) \subseteq (A(G_2)^*, A(\tilde{G}_2)^*) G_1) \}$$

is the image of $$G_1$$ under $$\phi$$ and

$$\phi^{-1}(G'_2) = \bigcup \{ \tilde{G}_1 \subseteq G_1 \mid \phi((A(\tilde{G}_1)^*, A(\tilde{G}_1)^*) G_1) \subseteq (A(G'_2)^*, A(G'_2)^*) G_1) \}$$

is the coimage of $$G'_2$$ under $$\phi$$.

Consider a fine homomorphism $$\phi: G_1 \rightarrow G_2$$. Then an admissible monoidhomomorphism $$h_\phi$$ is defined by $$h_\phi(\xi) = \phi(\xi) (\xi \in A(G_1))$$. We get $$G_{h_\phi} \subseteq G_2$$ and a factorizationdiagram

$$\begin{array}{c}
G_1 \\
\xrightarrow{\phi} \\
\cap G_2 \\
\xleftarrow{h_\phi}
\end{array}$$
In the case $h_{\phi}$ is $\varepsilon$-free, we get
\[ G_1 \subseteq G_{h_{\phi}} \]
and a factorization diagram
\[
\begin{array}{c}
G_1 \\
\phi \\
G_2
\end{array}
\begin{array}{c}
\subseteq \\
\phi_{h_{\phi}} \\
\phi_{h_{\phi}}
\end{array}
\begin{array}{c}
G_2
\end{array}
\]
Related to the concept of subgrammars is the concept of direct sums. [12]

Consider two grammars $G_1$ and $G_2$.
If $G_1 \cap G_2 = (\phi, \phi, \phi, \phi)$, we define $G_1 \oplus G_2 = G_1 \cup G_2$. With the inclusions $\nu_i \ (i=1, 2)$ we get a sum-diagram
\[
G_1 \xrightarrow{\nu_1} G_1 \oplus G_2 \xleftarrow{\nu_2} G_2.
\]
Observe that $\nu_i$ are very fine $\phi$-morphisms for $i=1, 2$.
Another interesting sum-diagram is obtained in the case $G_1 \cap G_2 = (T, \phi, T, \phi)$, where $T$ is a fixed alphabet. We define $G_1 + G_2 = G_1 \cup G_2$. With the inclusions $\nu_i \ (i=1, 2)$ we get a sum-diagram relative to $Z$-morphisms.
\[
G_1 \xrightarrow{\nu_1} G_1 + G_2 \xleftarrow{\nu_2} G_2.
\]
Observe that $\nu_i$ are very fine $\phi$-morphisms.

Remark.

i) $\mathcal{L}(G_1 + G_2) = \mathcal{L}(G_1) \cup \mathcal{L}(G_2)$,
ii) $\mathcal{L}(G_1 \oplus G_2) = \mathcal{L}(G_1) \cup \mathcal{L}(G_2)$.

Consider the following situation.
\[
\begin{array}{c}
G_1 \\
\phi_1 \\
G_1'
\end{array}
\begin{array}{c}
\nu_1 \\
\nu_2 \\
\phi_2
\end{array}
\begin{array}{c}
G_2 \\
\phi_1 \\
G_2'
\end{array}
\]
then there is a unique homomorphism
\[ \phi_1 \oplus \phi_2 : G_1 \oplus G_2 \rightarrow G_1' \oplus G_2' \]
with
\[ (\phi_1 \oplus \phi_2) \nu_i = \nu_i' \circ \phi_i \quad (i=1, 2) \]
Observe that properties of $\phi_i$ transfer to $\phi_1 \oplus \phi_2$, for example if $\phi_i$ are closed, then $\phi_1 \oplus \phi_2$ is closed.

We now turn our interest to products and factorizations.
Lemma 2.5. If $\phi: G_1 \rightarrow G_2$ is a fine homomorphism, then there is a commuting diagram

\[
\begin{array}{c}
G_3 \\
\phi_1 \downarrow \phi_2 \\
G_1 \xrightarrow{\phi} G_2
\end{array}
\]

such that $\phi_1$ is a fine $T$-morphism and $\phi_2$ is a very fine $Z$-morphism.

Proof. Consider the admissible monoidomorphism, defined by

\[
\begin{align*}
    h(\xi) &= \xi, \quad (\xi \in Z(G_3)), \\
    h(t) &= \phi(t), \quad (t \in T(G_1)), \\
    G_3 &= G_2 \quad \text{and} \quad \phi_1 = \phi_2.
\end{align*}
\]

We show

"$\phi_1(r) = \phi_1(r') \Rightarrow \phi(r) = \phi(r')$"

Let $r = (p, t_0\eta_1 \ldots \eta_k t_k)$ and $r' = (p', t_0'\eta_1' \ldots \eta_k' t_k')$, where $t_i, t_i' \in T(G_2)^*$, $\eta_i, \eta_i', p, p' \in Z(G_1)$.

Application of $\phi_1$ yields

\[
i = k, \quad p = p', \quad \eta_i = \eta_i'(1 \leq i \leq k) \quad \text{and} \quad \phi(t_i) = \phi(t_i')(0 \leq i \leq k).
\]

By this we get $\phi(r) = \phi(r')$.

With the above assertion the existence of $\phi_2$ follows immediately.

Corollary. If $\phi$ is closed, then $\phi_2$ is closed.

Theorem 2.2. To any pair $\phi_i: G_i \rightarrow G_0$ ($i = 1, 2$) of homomorphisms, where $\phi_1$ is fine, $\phi_2$ is very fine there exists a commuting diagram

\[
\begin{array}{c}
G_0 \\
\phi_1 \downarrow \\
G_1 \\
\phi_3 \\
\phi_4 \\
G_2 \\
\phi_4 \\
G_3
\end{array}
\]

where $\phi_4$ is fine and $\phi_3$ is very fine.

Furthermore

i) If $\phi_1$ is epic, then $\phi_4$ is epic.

ii) If $\phi_1$ is a $T$-morphism, then $\phi_4$ is a $T$-morphism.

iii) If $\phi_2$ is epic, then $\phi_3$ is epic.

Proof. The proof is in its essence a pullback-construction ([10, 13]) in three steps.

Step 1. Assume "$\phi_2$ is a $Z$-morphism". Define $G_3$ by

1) $Z(G_3) = \{(\xi, \xi') \in Z(G_2) \times Z(G_2) \mid \phi_1(\xi) = \phi_2(\xi')\}$, $T(G_3) = T(G_1)$,
2) $s(G_3) = \{(\sigma, \sigma') \in s(G_1) \times s(G_2) \mid \phi_1(\sigma) = \phi_2(\sigma')\}$.
2) \( P(G_3) = \{ (\eta_1, \eta_2, \ldots, \eta_s, \eta_0, t_0, \xi_1, \xi_2, \ldots, \xi_r, t_r) \mid s \geq 0, \ r \geq 0, \ t_i \in T(G_1)^* \ (0 \leq i \leq r), \)
\( (\eta_1, \ldots, \eta_s, t_0, \xi_1, \ldots, \xi_r, t_r) \in P(G_1) \) &
\( (\phi_1(\eta_1, \ldots, \eta_s), \phi_2(t_0, \xi_1, \ldots, \xi_r, t_r)) \in \phi_2(P(G_3)) \} \).

Now consider the monoid homomorphisms \( h_3 \) and \( h_4 \) defined by
\( h_3(\xi) = \xi, \quad h_4(\xi, \xi') = \xi' \),
\( h_3(t) = t, \quad h_4(t) = \phi_1(t) \quad (t \in T(G_1)) \).

These two monoid homomorphisms induce in a natural way homomorphisms \( \phi_3 \) and \( \phi_4 \) of the desired kind.

**Step 2.** Assume "\( \phi_2 \) is a \( T \)-morphism". Define \( G_3 \) by

1) \( Z(G_3) = Z(G_1), \quad T(G_3) = \{ (t, w) \mid t \in T(G_1), w \in T(G_2)^* \}, \phi_1(t) = \phi_2(w) \}, \)
\( s(G_3) = s(G_1) \),

2) \( P(G_3) = \{ (\phi, \eta_0(t_1, w_1) \ldots \eta_{s-1}(t_s, w_s) \eta_s) \mid s \geq 0, \ (t_i, w_i) \in T(G_1), \eta_i \in Z(G_1)^* \) &
\( (\phi, \eta_0 t_1 \ldots \eta_{s-1} t_s \eta_s) \in P(G_1) \) &
\( (\phi_1(\phi), \phi_2(\eta_0) \phi_3(w_1) \ldots \phi_1(\eta_{s-1}) \phi_2(w_s) \phi_1(\eta_s) \in \phi_2(P(G_3)) \} \).

Again, define monoid homomorphisms \( h_3 \) and \( h_4 \) by
\( h_3(\xi) = \xi, \quad h_4(\xi) = \phi_1(\xi) \quad (\xi \in Z(G_3)), \)
\( h_3(t, w) = t, \quad h_4(t, w) = w \quad (t, w) \in T(G_3)) \).

In a natural way homomorphisms \( \phi_3 \) and \( \phi_4 \) of the desired kind are induced.

**Step 3.** We combine Step 1 and Step 2 considering the given diagram

\[
\begin{array}{c}
G_1 \\
\downarrow \phi_1 \\
G_0 \leftarrow \phi_2 \rightarrow G_2
\end{array}
\]

By the factorization-lemma 2.5 we get the diagram

\[
\begin{array}{c}
G_1 \\
\downarrow \phi_1 \\
G_0 \leftarrow \phi_2 \\
\downarrow \phi_3 \\
G_2
\end{array}
\]

where \( \psi_1 \) is a \( Z \)-morphism and \( \psi_2 \) is a \( T \)-morphism.

By Step 1 we get a diagram

\[
\begin{array}{c}
G_1 \leftarrow \phi_3 \rightarrow G'_3 \\
\downarrow \phi_1 \\
G_0 \\
\downarrow \phi_2 \\
G_2
\end{array}
\]

\[
\begin{array}{c}
G_1 \leftarrow \phi_2 \rightarrow G'_3 \\
\downarrow \phi_1 \\
G_0 \\
\downarrow \phi_2 \\
G_2
\end{array}
\]
By Step 2 we get a diagram

\[
\begin{array}{c}
G_1 \xleftarrow{x_2} G_3 \xleftarrow{x_2} G_3 \\
\phi_1 \\
\downarrow z_1 \\
G_0 \xleftarrow{\psi_1} G_2 \\
\end{array}
\]

Now the statement of our theorem follows immediately.

3. Grammar Forms

Consider a grammar \(G\). Following Ginsburg et al. [3], we define an interpretation \(\mu\) of \(G\) to be a substitution \(\mu: A(G)^* \rightarrow 2^{A(G')^*}\) where \(G'\) is another grammar such that the following conditions hold:

i) \(\xi \in Z(G) \Rightarrow \mu(\xi) \subseteq Z(G')\),

ii) \(s(G') \subseteq \mu(s(G))\),

iii) \(\xi = \xi' \in Z(G) \Rightarrow \mu(\xi) \cap \mu(\xi') = \phi\),

iv) \(t \in T(G) \Rightarrow \mu(t) \subseteq T(G')^* \& \mu(t)\) is finite,

v) \(P(G') \subseteq \{ (\phi', q') | (\phi, q) \in P(G) : \phi' \in \mu(\phi) \text{ and } q' \in \mu(q) \}\).

We use the notion \(\mu: G \Rightarrow G'\) for interpretations.

**Theorem 3.1.** The following statements are equivalent.

i) There exists \(\mu: G \Rightarrow G'\).

ii) There exists a diagram

\[
\begin{array}{c}
G \xleftarrow{\phi_1} G'' \xrightarrow{\phi_2} G' \\
\end{array}
\]

where \(\phi_1\) is very fine and \(\phi_2\) is an epic and fine \(T\)-morphism.

**Proof.** i) \(\Rightarrow\) ii): Consider \(T = \{ (t, t') | t' \in \mu(t), t \in T(G) \}\).

Define \(h: (T \cup Z(G'))^* \rightarrow A(G)^*\) by

\[
h(t, t') = t \\
h(\xi) = \eta \quad (\xi \in Z(G'), \xi \in \mu(\eta)).
\]

Obviously \(h\) is admissible and well-defined. Furthermore define \(h_1: (T \cup Z(G'))^* \rightarrow A(G')^*\) by

\[
h_1(t, t') = t' \& h_1(\xi) = \xi \quad (\xi \in Z(G')).
\]

Now consider \(\phi_{h_1}\) and \((G^h)_{h_1}\).

By this we get the diagram

\[
\begin{array}{c}
G \xleftarrow{\phi^h} G^h \xrightarrow{\phi_{h_1}} (G^h)_{h_1} \cong G'. \\
\end{array}
\]
Then there exists a diagram

\[
G \xleftarrow{\phi^h} G^h \xrightarrow{\phi_{h_1}} (G^h)_{h_1} \\
\phi_{h_1}^{-1} \quad \text{UL} \quad \text{UL} \\
(G^h) \quad \text{UL} \\
\phi_{h_1} \quad \text{UL} \\
G' \xleftarrow{\phi_2} G'' \xrightarrow{\phi_1} G'
\]

which gives the desired diagram of ii).

ii) $\Rightarrow$ i): Consider the diagram

\[
G \xleftarrow{\phi_1} G'' \xrightarrow{\phi_2} G'.
\]

Since $\phi_1$ is very fine there exists a length-preserving and admissible $h$ such that

\[
(G^h) \xleftarrow{\phi_{h_1}} G'' \xrightarrow{\phi_1} G'
\]

commutes.

Furthermore there exists an admissible $h_1'$ such that the diagram

\[
G' \xleftarrow{\phi_2} G'' \xrightarrow{\phi_{h_1'}} (G^h)_{h_1'} \\
\phi_{h_1'} \quad \text{UL} \\
(G^h)_{h_1'} \quad \text{UL} \\
\phi_{h_1'} \quad \text{UL} \\
G' \xleftarrow{\phi_2} G''
\]

commutes.

Hence we have reached the following situation

\[
(G)^h \xrightarrow{\text{UL}} \left( (G)^h \right)_{h_1'} \\
\phi_{h_1'} \quad \text{UL} \\
\left( (G)^h \right)_{h_1'} \quad \text{UL} \\
\phi_{h_1'} \quad \text{UL} \\
G' \xleftarrow{\phi_2} G''
\]

A diagram

\[
G \xleftarrow{\phi^h} (G)^h \xrightarrow{\phi_{h_1'}} (G)^h_{h_1'} \cong G'
\]

results with $h_1'/Z((G)^h)$ is the identity.

Now define the interpretation $\mu$ by

\[
\mu(\eta) = h^{-1}(\eta) \quad (\eta \in Z(G))
\]

and

\[
\mu(t) = h_1'h^{-1}(t) \quad (t \in T(G)).
\]

We are now in the position to give our definition of grammar forms.

**Definition 3.1.** A grammarform $\Gamma$ is a triple $\Gamma = (G, H_1, H_2)$, where $G$ is a grammar and $H_1, H_2$ are classes of homomorphisms.
If $\Gamma'$ is a grammar form we associate a grammar family $|\Gamma'|$ by defining
$|\Gamma'| = \{G_1\}$ there exists a diagram $G \leftarrow G_2 \rightarrow G_1$ with $\phi_1 \in H_1, \phi_2 \in H_2$

If $\mathcal{G}$ is a grammar family then
$\mathcal{L} (\mathcal{G}) = \{L | \exists G \in \mathcal{G} : \mathcal{L}(G) = L\}$

is the associated language family.

In case $\mathcal{G} = |\Gamma'|$ we write $\mathcal{L} (\mathcal{G}) = \mathcal{L} (\Gamma)$.

Definition 3.2. A grammarform $\Gamma' = (G, H_1, H_2)$ is regular if and only if
i) $\phi$ isomorphism $\Rightarrow \phi \in H_1 \cap H_2$,
ii) $\phi_1, \phi_2 \in H_1 \Rightarrow \phi_1 \circ \phi_2 \in H_1$,
iii) $\phi_1, \phi_2 \in H_2 \Rightarrow \phi_1 \circ \phi_2 \in H_2$,
iv) To any pair $\phi_i : G_i \rightarrow G_0$ ($i = 1, 2$) with $\phi_1 \in H_2$ and $\phi_2 \in H_1$ there exists a commuting diagram

$$
\begin{array}{ccc}
G_1 & \xleftarrow{\phi_3} & G_3 \\
\downarrow & & \downarrow \\
G_0 & \xleftarrow{\phi_2} & G_2 \\
\end{array}
$$

with $\phi_3 \in H_1$ and $\phi_4 \in H_2$.

The most significant axiom is "iv)". Theorem 2.2 is an example for this axiom.

Lemma 3.1. If $\Gamma' = (G, H_1, H_2)$ is a regular grammar form, $G' \in |\Gamma|$, then $\Gamma'' = (G', H_1, H_2)$ is regular and $|\Gamma''| \subseteq |\Gamma|$.

Proof. We have to show the inclusion. Consider a fixed diagram

$G \leftarrow G'' \rightarrow G', \quad \phi_1 \in H_1 \& \phi_2 \in H_2$

for $G'$ and an arbitrary diagram

$G' \leftarrow G_1 \rightarrow G_1, \quad \phi_1 \in H_1, \quad \phi_2 \in H_2$.

By axiom iv) there exists a diagram

$G \leftarrow G'' \rightarrow G_3 \leftarrow G_1 \rightarrow G_1$

where $\phi_3 \in H_1$ and $\phi_4 \in H_2$.

Now by the axioms ii) and iii) we obtain $G_1 \in |\Gamma|$, and this proves the lemma.

Corollary. If $\Gamma = (G, H_1, H_2)$ and $\Gamma'' = (G', H_1, H_2)$ are regular grammar forms then:

$|\Gamma| = |\Gamma''| \Leftrightarrow G' \in |\Gamma| \& G \in |\Gamma''|$

We call a homomorphism $\phi : G_1 \rightarrow G_2$ $\varepsilon$-free if $\phi(w) \neq \varepsilon$ for all $w \in T(G_1)$, $w \neq \varepsilon$. 
Theorem 3.2. Let $\Gamma = (G, H_1, H_2)$ be a regular grammar form.

1) If all very fine and epic Z-morphisms $\phi$ are in $H_1$, then $\mathcal{L}(\Gamma)$ is closed under intersection with regular sets.

2) If all fine and $\varepsilon$-free and epic $T$-morphisms are in $H_2$, then $\mathcal{L}(\Gamma)$ is closed under $\varepsilon$-free homomorphisms.

3) If all very fine and epic $T$-morphisms are in $H_2$, and all very fine and epic homomorphisms are in $H_1$, then $\mathcal{L}(\Gamma)$ is closed under union if $H_1$ and $H_2$ are closed under the operation $\oplus$.

Proof. 1) Consider $L = \mathcal{L}(G'), G' \in \mathcal{I}$, and $R$ regular, then there exists a diagram

$$G \leftarrow \phi_1 \leftarrow G' \leftarrow \psi_1 \leftarrow G_R$$

where $\phi_1 \in H_1$, $\phi_2 \in H_2$, and $\psi_1 \in H_1$.

By axiom iv) we can "fill-in" this diagram and get

$$G \leftarrow \phi_1 \in H_1 \leftarrow G' \leftarrow \phi_2 \in H_2 \leftarrow G_R.$$

Now $\phi_1, \phi_2 \in H_1$ and 1) is following.

2) Consider an $\varepsilon$-free homomorphism $h$ and $G' \in \mathcal{I}$, then we get a diagram

$$G \leftarrow \phi_1 \in H_1 \leftarrow G' \leftarrow \phi_2 \in H_2 \leftarrow G_R.$$

Since $\phi_1 \in H_2$ we get the assertion by observing $\phi_1 \phi_2 \in H_2$.

3) First we observe the following fact

"If $G$ is a grammar, $T \supseteq T(G)$ an alphabet, then there exists a grammar $G'$ with $T(G') = T, L(G') = L(G)$ and a very fine and epic $T$-morphism $\phi: G' \rightarrow G'$.

This fact means, we can always add unnecessary terminals. Now, consider two arbitrary diagrams

$$G \leftarrow \phi_1 \in H_1 \leftarrow G_1 \leftarrow \phi_2 \in H_2 \leftarrow G_2,$$

$$G \leftarrow \phi_1 \in H_1 \leftarrow G_2 \leftarrow \phi_2 \in H_2 \leftarrow G_2.$$

By our observation we can assume $T(G_1) = T(G_2) = T$. By axiom i) we can find

$$\tilde{G} \supseteq G, \quad \tilde{G}_1 \supseteq G_1, \quad \tilde{G} \supseteq G, \quad \tilde{G}_1 \supseteq G_1,$$

such that $T(\tilde{G}_1) = T(G_1)$ and

$$G \oplus \tilde{G}, \quad G_1 \oplus \tilde{G}_1, \quad G_1 \oplus G_2, \quad \tilde{G}_1 \oplus G_2$$

are defined, and $\tilde{G}_i (i = 0, 1, 2, 3)$ are all very fine and epic $T$-morphisms.
Moreover we can find a very fine and epic $T$-morphism $\chi': \bar{G}_1 \oplus G_2 \rightarrow \bar{G}_1 + G_2$
and a very fine and epic homomorphism $\chi: \bar{G} \oplus G \rightarrow G$.

Then we get the following diagram

$$
\begin{array}{cccc}
\bar{G} & \xleftarrow{\bar{\phi}_1} & \bar{G}_1' & \xrightarrow{\bar{\phi}_2} & \bar{G}_1 \\
\downarrow{\bar{\phi}_1} & & \downarrow{\bar{\phi}_1} & & \\
G & \xleftarrow{\phi_1} & G_1' \oplus G_2 & \xrightarrow{\phi_2 \oplus \phi_2} & \bar{G}_1 + G_2 \\
\downarrow{\phi_1} & & \downarrow{\phi_2} & & \\
G & \xleftarrow{\phi_1} & G_2 & \xrightarrow{\phi_2} & G_2
\end{array}
$$

where $\bar{\phi}_1, \phi_1 \in H_1$ (axioms i) and ii) and $\bar{\phi}_2, \phi_2 \in H_2$ (axioms i) and iii).

Now our assumption on the operation ' $\oplus$ ' and the axioms ii) and iii) give the desired result.

One of the main points of our proofs is the possibility of dualization by changing the direction of arrows. If $I' = (G, H_1, H_2)$ is a grammar form, then the cofamilies $|I'|^\omega$ is defined by

$$|I'|^\omega = \{G_1 | \text{There exists a diagram } G \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_1 \text{ where } \phi_1 \in H_1 \text{ and } \phi_2 \in H_2\}. $$

Analogously, $\mathcal{L}_c(I') = \mathcal{L}_c(|I'|^\omega)$ is the associated language cofamily.

In dualizing we define "coregular" be leaving axioms i), ii) and iii) unchanged, and dualizing axiom iv) in the following way axiom iv)$^\omega$: To any pair $\phi_i: G_0 \rightarrow G_i$ ($i = 1, 2$) with $\phi_1 \in H_2$ and $\phi_2 \in H_1$ there exists a commuting diagram

$$
\begin{array}{cccc}
G_1 & \xleftarrow{\phi_3} & G_3 & \\
\downarrow{\phi_1} & & \downarrow{\phi_4} & \\
G_0 & \xleftarrow{\phi_2} & G_2
\end{array}
$$

with $\varepsilon \in H_1$ and $\phi_4 \in H_2$.

**Remark.** Examples for axiom iv)$^\omega$ can be given by using congruence lattices [10].

By dualization we get the following lemmas and theorems, where the duality is denoted by the superscript "$^\omega$".

We indicate by two proof-examples how dualization should work.

**Lemma 3.1**. If $I' = (G, H_1, H_2)$ is a coregular grammar form, $G' \in |I'|^\omega$ then $I'' = (G', H_1, H_2)$ is coregular and $|I''|^\omega \subseteq |I'|^\omega$.

**Proof.** Consider a fixed diagram

$$
G \xrightarrow{\phi_1} G' \xrightarrow{\phi_2} G'' \xleftarrow{\phi_3} G' \quad \text{with } \phi_1 \in H_1 \& \phi_2 \in H_2
$$
and an arbitrary diagram
\[
G' \xrightarrow{\phi_1} G'_1 \xleftarrow{\phi_1'} G_1 \quad \text{with } \phi_1', \phi_2' \in H_2.
\]
By axiom \text{iv} there exists a diagram
\[
\begin{array}{c}
G \xrightarrow{\phi_1} G'' \xleftarrow{\phi_2} G' \\
\downarrow \quad \downarrow \phi_1 \\
G_1 \xleftarrow{\phi_1, \phi_2} G_1' \xleftarrow{\phi_2'} G_1.
\end{array}
\]
Now by axiom ii) and iii) we get \(G_1 \in |I'|^\infty\), which proves the lemma.

\textbf{Corollary}. If \(I = (G, H_1, H_2)\) and \(I'' = (G', H_1, H_2)\) are coregular grammar forms then
\[
|I''|^\infty = |I'|^\infty \Rightarrow G' \in |I''|^\infty \& G \in |I''|^\infty.
\]

\textbf{Theorem 3.2}. Let \(I = (G, H_1, H_2)\) be a coregular grammar form.

1) If all very fine and epic \(Z\)-morphisms are in \(H_2\), then \(L^\infty(I)\) is closed under intersection with regular sets.

2) If all fine and \(\varepsilon\)-free \(T\)-morphisms are in \(H_1\), then \(L^\infty(I)\) is closed under \(\varepsilon\)-free homomorphisms.

3) If all very fine and epic \(T\)-morphisms are in \(H_1\) and all very fine and epic homomorphisms are in \(H_1\), then \(L^\infty(I)\) is closed under union provided \(H_1\) and \(H_2\) are closed under the operation ‘\(\oplus\)’.

\textbf{Proof}. 1) Follows immediately by axiom iii).

2) Consider a diagram
\[
\begin{array}{c}
G \xrightarrow{\phi_1} G'' \xleftarrow{\phi_1'} G' \xrightarrow{\phi_2} G_1
\end{array}
\]
with \(\phi_1 \in H_1, \phi_2 \in H_2, \phi_0 \) fine and \(\varepsilon\)-free \(T\)-morphism. Since \(\phi_0 \in H_1\) we get by axiom \text{iv}) a diagram
\[
\begin{array}{c}
G \xrightarrow{\phi_1} G'' \xleftarrow{\phi_1'} G' \\
\downarrow \phi_0 \quad \downarrow \phi_0 \\
G_1 \xleftarrow{\phi_2} G_1
\end{array}
\]
where \(\phi_0 \in H_1, \phi_0' \in H_2\). Now by axiom ii) we get \(G_1 \in |I''|^\infty\).

3) Dualize part 3) of the proof of Theorem 3.2, where the sum-diagram is considered to be self-dual.

\textbf{Notation}. If \(I = (G, H_1, H_2)\), then \(I^\infty = (G, H_2, H_1)\).

\textbf{Lemma 3.2}. If \(I\) is regular then
\[
|I^\infty|^\infty \subseteq |I|^\infty.
\]
Proof. Consider a diagram $G \xrightarrow{\phi_1 \in \mathcal{H}_1} G' \xleftarrow{\phi_2 \in \mathcal{H}_2} G''$, then by axiom iv) we can fill in this diagram by

\[
\begin{array}{c}
G \\ \downarrow \phi_1 \in \mathcal{H}_1 \\
G' \\ \downarrow \phi_2 \in \mathcal{H}_2 \\
G'' \\
\end{array}
\xrightarrow{\phi_3 \in \mathcal{H}_1} G'' \\
\xleftarrow{\phi_4 \in \mathcal{H}_2} G'
\]

But this proves $G' \in \mathcal{I}$. Dualizing yields

**Lemma 3.2.** If $\mathcal{I}$ is coregular, then

\[ |\mathcal{I}^{\circ} | \subseteq |\mathcal{I}^{\circ \circ} |. \]

**Corollary.** If $\mathcal{I}$ is regular and $\mathcal{I}^{\circ} \text{ coregular}$, then

\[ |\mathcal{I}| = |\mathcal{I}^{\circ \circ} |. \]

**Proof.** Observe

\[ |\mathcal{I}| = |(\mathcal{I}^{\circ \circ})^{\circ}| \subseteq |\mathcal{I}^{\circ \circ} |. \]

We now turn our interest to grammar indices which have been introduced by A. Salomaa [9].

Consider a grammar $G$ and a derivation $f$. Let $\Delta (f)$ be the set of all sequential representations of $f$. To any $\delta \in \Delta (f)$,

\[ \delta = (1_w \times r_1 \times 1_v) \circ \cdots \circ (1_w \times r_s \times 1_v) \]

we assign

\[ \# (\delta) = \text{Max} \{ |u_i \circ (r_i) \circ v_i| | \ 1 \leq i \leq s \} \]

where $|w|_z$ is the number of variables in $w$.

Then we can assign to $f$ a number

\[ \# (f) = \text{Min} \{ \# (\delta) | \ \delta \in \Delta (f) \} \]

and define the index of $G$ to be

\[ \text{index} (G) = \text{Sup} \{ \# (f) | f \in D (G) \} \]

We study the behaviour of the index function under homomorphisms.

**Lemma 3.3.** i) $G_1 \subseteq G_2 \Rightarrow \text{index} (G_1) \leq \text{index} (G_2)$.

ii) If $\phi: G_1 \rightarrow G_2$ is fine, then $\text{index} (G_1) \leq \text{index} (G_2)$.

**Proof.** i) and ii) is trivial:

i) We have to show:

\[ \text{index} (\phi (f)) = \text{index} (f). \]

Since $\phi$ is on variables length preserving the above assertion follows from the fact, that $\phi$ induces a surjective mapping from $\Delta (f)$ onto $\Delta (\phi (f))$. 
But this is true, since all sequential representations of \( f \) can be generated by applying successively the following rules to a fixed sequential representation

\[
((1_u \times r \times 1_{v_1} c(r) v_2)) \circ (1_{u_3} r_1 \times r' \times 1_{v_3}) \equiv (1_{u_2} r_1 \times r' \times 1_{v_3}) \circ (1_u \times r \times 1_{v_3})
\]

and

\[
(1_{u_4} r_1 \times r \times 1_{v_3}) \circ (1_{u_5} r_1 \times r' \times 1_{v_3}) \equiv (1_{u_6} r_1 \times r' \times 1_{v_3}) \circ (1_{u_7} d(r) u_8 \times r' \times 1_{v_3})
\]

and these rules are transformed into the corresponding rules via \( \phi \).

Now we can prove a generalization of a theorem of [3].

**Theorem 3.3.** Consider a grammar form \( \Gamma = (G, H_1, H_2) \) where

i) \( \phi \in H_1 \Rightarrow \phi \) is fine,

ii) \( \phi \in H_2 \Rightarrow \phi \) is closed and fine,

iii) index \((G) < \infty\),

then \( \mathcal{L}(\Gamma) \) is contained in the class of nonexpansive languages [4].

**Proof.** By Lemma 3.3 we know

\( G' \in |\Gamma| \Rightarrow \text{index } (G') \leq \text{index } (G) < \infty. \)

Now a theorem of A. Salomaa [9] states:

If \( G_1 \) is a grammar

\[
\text{Sup Inf } \left\{ \overrightarrow{d}(f) \mid d(f) \in \text{s}(G_1 \& c(f) = w) < \infty \right\} < \infty \quad w \in \mathcal{L}(G_1)
\]

then \( \mathcal{L}(G_1) \) is nonexpansive.

But this proves our theorem.

Again we have a dual version of the theorem.

**Theorem 3.3**. Consider a grammar form \( \Gamma = (G, H_1, H_2) \) where

i) \( \phi \in H_1 \Rightarrow \phi \) is closed and fine,

ii) \( \phi \in H_2 \Rightarrow \phi \) is fine,

iii) index \((G) < \infty\),

then \( \mathcal{L}(\Gamma) \) is contained in the class of nonexpansive languages.

**References**


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