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Grammarforms and Grammarhomomorphisms

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Summary. We show that grammar forms, introduced by S. Ginsburg, can be treated with the help of grammar homomorphisms, introduced by G. Hotz. This gives us the possibility to generalize the notion of grammar forms in such a way that we can prove closure properties of the associated language family using some simple axioms by diagram chasing. Moreover we demonstrate a principle of duality in dealing with this theory.

Introduction

There has been an extensive study of families of languages and automata. In contrast to this fact, there is no appropriate theory of grammar families. Recently, Ginsburg et al. [3], introduced a theory of grammar families using grammarforms. By this approach grammar families are generated via interpretations of a given grammar, called grammarform, which reflects the structural properties of the productions of the interpreted grammar.

The aim of this paper is to demonstrate how this approach can be embedded in the theory of grammar homomorphisms, introduced by G. Hotz [5, 6, 8].

We'll show that the mechanism of grammar forms can be generalized in a natural way such that closure properties of the generated language family can be derived by diagram chasing. Moreover this generalization demonstrates a principle of duality.

1. Grammars and X-Categories

We give some basic definitions and a short review of the construction of X-categories ([5, 8]).

A *semi-Thue-system* is a pair $S=(A(S), P(S))$ such that $A(S)$ is an alphabet and $P(S)$ is a finite subset of $A(S)^* \times A(S)^*$ where $A(S)^*$ is the free monoid over $A(S)$.

Free X-categories are precise definitions of "derivations" associated with semi-Thue-systems.

Let $w, v \in A(S)^*$: w is *derivable* into $v\,(w \vdash v)$ (with respect to S), if there exists a sequence.

$$w=u_0 p_0 q_0, \quad u_0 q_0 v_0 = u_1 p_1 v_1, \ldots, \quad u_{r-1} q_{r-1} v_{r-1} = u_r p_r v_r, \quad u_r q_r v_r = v$$

where $r \geqq 0$, $u_i, v_i, p_i, q_i \in A(S)^*$ $(0 \leqq i \leqq r)$ and $(p_i, q_i) \in P(S)$ $(0 \leqq i \leqq r)$.

This sequence is called a *derivation*. The set of all derivations defines a free X-category $F(S)$. Informally $F(S)$ can be described as follows:

the objects are the elements of $A(S)^*$,
the morphisms are the "derivations",
if $f=(w \vdash v)$ is a derivation, then
$df=w$ is the *domain* of f and
$cf=v$ is the *codomain*,
the empty derivations ($r=0$) are the identities 1_w,
the operation "$\circ$" is the concatenation of derivations illustrated by the diagram

$$(w \overset{f}{\vdash} v \overset{g}{\vdash} z) = g \circ f$$

the monoidal operation "$\times$" is illustrated by the diagram

$$\begin{bmatrix} w & \overset{f}{\vdash} v \\ w_1 & \underset{g}{\vdash} v_1 \end{bmatrix} = f \times g.$$

Note, that in a natural way $P(S)$ can be embedded into the set of morphisms of $F(S)$, such that $d(p, q)=p$ and $c(p, q)=q$.

A *functor* between X-categories is a covariant functor, which is a monoid-homomorphism with respect to the monoidal operation. A functor between $F(S)$ and another X-category $\mathfrak{X}$ is now uniquely determined by exhibiting a monoid-homomorphism ϕ_0 between the objects and a mapping ϕ'_M from $P(S)$ into the morphism-set of $\mathfrak{X}$, such that

$$c, d(\phi'_M(r)) = \phi_0(c, d(r))$$

for all $r \in P(S)$.

We adopt some further notations.

If f is a morphism of $F(S)$, we write $f \in F(S)$. Now, consider two subsets L_1 and L_2 of $A(S)^*$, then

$$(L_1, L_2)_S = \{f \in F(S) \mid df \in L_1 \& cf \in L_2\}.$$

If $f \in F(S)$ then f can be represented in the form

$$f = (1_{w_1} \times r_1 \times 1_{v_1}) \circ \cdots \circ (1_{w_s} \times r_s \times 1_{v_s})$$
$$(s \geqq 0,\ w_i, v_i \in A(S)^*,\ r_i \in P(S)\ \ 1 \leqq i \leqq s).$$

This representation is called a *sequential* representation of f. The number s is independent a special representation of f. It is called the *length* $\|f\|$ of f.

Obviously,

i) $\|f\|=0 \Leftrightarrow f$ is an identity,
ii) $\|f \circ g\| = \|f\| + \|g\|$,
iii) $\|f \times g\| = \|f\| + \|g\|$.

For further details the reader is referred to [8]. A *grammar* G is a 4-tuple

$$G = (A(G), P(G), T(G), s(G))$$

where

i) $(A(G), P(G))$ is a semi-Thue-system,

ii) $P(G) \subseteq Z(G)^{+} \times A(G)^{*}$ where
$Z(G) = A(G) \setminus T(G)$ and $Z(G)^{+} = Z(G)^{*} \setminus \{\varepsilon\}$,

iii) $T(G) \cup s(G) \subseteq A(G)$ & $s(G) \subseteq Z(G)$.

As usual, $A(G)$ is the *alphabet*, $T(G)$ the set of *terminals*, $Z(G)$ the set of *variables*, $s(G)$ the set of *axioms* and $P(G)$ the set of *productions* of G. With respect to X-categories we transfer our notation from semi-Thue-systems to grammars. Then

$$D(G) = (s(G), T(G)^{*})_{G}$$

is the set of *derivations of* G, and

$$\mathscr{L}(G) = c(D(G))$$

is the *generated language*.

We assume that the reader is familiar with standard notions of the language theory [2].

2. Grammar Homomorphisms

We discuss algebraic concepts related to the use of X-categories.

A more extensive study can be found in [1] and [11]. Consider two grammars G_1 and G_2.

A *homomorphism* $\phi: G_1 \to G_2$ is a functor $\phi: F(G_1) \to F(G_2)$ with

i) $\phi(T(G_1)) \subseteq T(G_2)^{*}$,

ii) $\phi(s(G_1)) \subseteq s(G_2)$,

iii) $\phi(Z(G_1)) \subseteq Z(G_2)$.

Remark. If $\phi: G_1 \to G_2$ is a homomorphism, then $\phi(D(G_1)) \subseteq D(G_2)$ and therefore $\phi(\mathscr{L}(G_1)) \subseteq \mathscr{L}(G_2)$.

We now briefly discuss various constructions, which can serve as examples for special classes of homomorphisms.

Consider triples (Z, T, s) of alphabets such that $Z \cap T = \phi$ and $s \subseteq Z$.

A monoidhomomorphism

$$h: (Z_1 \cup T_1)^{*} \to (Z_2 \cup T_2)^{*}$$

is called *admissible* if

i) $h(T_1) \subseteq T_2^{*}$,

ii) $h(s_1) \subseteq s_2$ and

iii) $h(Z_1) \subseteq Z_2$.

If G is a grammar,

$$h: (Z(G), T(G), s(G)) \to (Z, T, s)$$

is admissible, then a grammar G_h and a homomorphism $\phi_h\colon G \to G_h$ are defined by

i) $Z(G_h)=Z, \quad s(G_h)=s, \quad T(G_h)=T,$

ii) $P(G_h)=\{(h(p), h(q)) \mid (p, q) \in P(G)\},$

iii) ϕ_h is on objects equal to h, and
$\phi_h(p, q)=(h(p), h(q)) \ ((p, q) \in P(G)).$

Fact. $\phi_h(P(G))=P(G_h)$.

This observation leads to the following definition.

Definition 2.1. $\phi\colon G_1 \to G_2$ a homomorphism.

i) ϕ is a *fine* homomorphism iff $\phi(P(G_1)) \subseteq P(G_2)$,

ii) ϕ is a *very fine* homomorphism iff $\phi(T(G_1)) \subseteq T(G_2)$ & ϕ is fine.

We can prove more.

Lemma 2.1. If h is an admissible ε-free monoidhomomorphism then $\phi_h(D(G)) = D(G_h)$ and therefore $h(\mathscr{L}(G)) = \mathscr{L}(G_h)$, if h is injective on variables and $h^{-1}(s(G_h)) = s(G)$.

Proof. We consider the set

$$\hat{D}(G)=\{f \in F(G) \mid \quad \|f\| \geqq 1,\ df \in s(G)\ cf \in A(G)^*\}$$

and the corresponding set $\hat{D}(G_h)$.

If we can show that $\phi_h(\hat{D}(G))=\hat{D}(G_h)$ we get the desired result.

We therefore show by induction on $\|f'\|$, that to any $f' \in \hat{D}(G_h)$ there exists $f \in \hat{D}(G)$ with $\phi_h(f)=f'$. "$\|f'\|=1$":

In this case

$$f'=(\sigma, q) \in P(G_h) \quad \text{with } \sigma \in s(G_h).$$

Since $\phi_h(P(G))=P(G_h)$ we find $(\sigma', q') \in P(G)$ with $\phi_h((\sigma', q'))=(\sigma, q)$.

Moreover $\sigma' \in h^{-1}(\sigma) \subseteq h^{-1}(s(G_h))=s(G)$.

$$\text{"}\|f'\|=k \Rightarrow \|f'\|=k+1\text{"}:$$

We consider a decomposition $f'=(1_{u'} \times r' \times 1_{v'}) \circ f_1'$ with $r' \in P(G_h)$ and $f_1' \in \hat{D}(G_h)$.

By the induction hypothesis there exists $f_1 \in \hat{D}(G)$ with

$$f_1'=\phi_h(f_1), \quad c(f_1)=upv \quad \text{and} \quad h(u)=u', \quad h(v)=v'.$$

On the other hand there exists a production $r=(p', q')$ with $\phi_h(r)=r'$.

Now we know

$$h(p)=d(r')=h(p').$$

Since h is injective on variables we get $p=p'$; but then $\phi((1_u \times r \times 1_v) \circ f_1)=f'$, which proves our lemma.

Definition 2.2. Let be $\phi\colon G_1 \to G_2$ a homomorphism.

i) ϕ is *epic* iff $\phi(P(G_1))=P(G_2)$,

ii) ϕ is *closed* iff $\phi(D(G_1))=D(G_2)$.

A modification of the proof of Lemma 2.1 yields

Lemma 2.2. If $\phi: G_1 \to G_2$ is a fine homomorphism, with ϕ injective on variables, then ϕ is closed, if $\phi^{-1}(s(G_2)) = s(G_1)$.

Consider in contrast to the above construction the following situation. Let G a grammar and

$$h: (Z, T, s) \to (Z(G), T(G), s(G))$$

admissible. We assume, h to be ε-free, then we can define a grammar G^h by

i) $Z(G^h) = Z$, $T(G^h) = T$, $s(G^h) = s$,

ii) $P(G^h) = \{(p, q) \mid (h(p), h(q)) \in P(G)\}$.

Note, that in this case $(G^h)_h = G$; therefore, there is a fine homomorphism $\phi^h: G^h \to G$.

The homomorphisms ϕ^h are special cases of another type of closed homomorphisms, introduced by G. Hotz [7], [6].

Definition 2.3. Let be $\phi: G_1 \to G_2$ a homomorphism.

i) ϕ is *normal* if and only if
$(p, q) \in P(G_1)$, $p' \in A(G_1)^*$,
$\phi(p) = \phi(p')$ implies $(p', q) \in P(G_1)$,

ii) ϕ is *conormal* if and only if
$(p, q) \in P(G_1)$, $q' \in A(G_1)^*$,
$\phi(q) = \phi(q')$ implies $(p, q') \in P(G_1)$.

Fact. ϕ_h is both normal and conormal and epic.

Lemma 2.3. If $\phi: G_1 \to G_2$ is epic and normal, then ϕ is closed, if

$$\phi^{-1}(s(G_2)) = s(G_1).$$

Proof. Without loss of generality we can assume ϕ to be epic. We show by induction on the length of derivations, that to any $f' \in \hat{D}(G_2)$ there exists $f \in \hat{D}(G_1)$ with $\phi(f) = \phi(f')$.

If $\|f'\| = 1$, there is nothing to prove.

Now assume

$$f' = (1_{u'} \times r' \times 1_{v'}) \circ f_1'.$$

Then there exists a derivation $f_1 \in \hat{D}(G_1)$ with $\phi(f_1) = f_1'$. Since ϕ is epic there exists $r \in P(G_1)$ with $\phi(r) = r'$. We know

$$c f_1 = u p v$$

with

$$\phi(u) = u', \quad \phi(v) = v', \quad \phi(p) = \phi d(r) = d(r').$$

Since ϕ is normal, we get $(p, c(r)) \in P(G_1)$.

Moreover

$$\phi((p, c(r))) = (\phi(p), \phi(c(r))) = \phi(r) = r'.$$

By this we get

$$\phi((1_u \times (p, c(r)) \times 1_v) \circ f_1) = f'.$$

In the case of conormal homomorphisms we can show more

Lemma 2.4. If $\phi: G_1 \to G_2$ is conormal and epic, then ϕ is closed and $\phi(\mathscr{L}(G_1)) = \mathscr{L}(G_2)$ if $\phi^{-1}(s(G_2)) = s(G_1)$.

Proof. We show:

To any $w \in A(G_1)^*$ such that there exists a derivation $f' \in D(G_2)$ with $cf' = \phi(w)$, there exists a derivation $f \in D(G_1)$ with $\phi(f) = f'$ and $cf = w$. Again we apply induction on the length of f'.

If $\|f'\| = 1$, then $(\sigma', \phi(w)) \in P(G_2)$ with $\sigma' \in s(G_2)$. But then there exists $(\sigma, w_1) \in P(G_1)$ with $\phi(\sigma) = \sigma'$ and $\phi(w_1) = \phi(w)$. Since ϕ is conormal $(\sigma, w) \in P(G_1)$.

If $\|f'\| > 1$, we consider a decomposition

$$f' = (1_{u'} \times (p', q') \times 1_{v'}) \circ f'_1.$$

We know $w = uqv$ with $\phi(u) = u'$, $\phi(q) = q'$ and $\phi(v) = v'$. Then there is a production $(p, \tilde{q})$ with $\phi(p) = p'$ and $\phi(\tilde{q}) = q'$.

Since ϕ is conormal we get $r = (p, q) \in P(G_1)$ and $\phi(r) = (p', q')$.

Now consider the word upv; then $\phi(upv) = d(f'_1)$. By the induction hypothesis there is a derivation $f_1 \in \widehat{D}(G_1)$ with $\phi(f_1) = f'_1$ and $c(f_1) = upv$. But then we can build

$$f = (1_u \times (p, q) \times 1_v) \circ f_1$$

which gives the desired result.

Examples of very fine homomorphisms ϕ with

i) ϕ is surjective on objects,

ii) ϕ is epic,

iii) ϕ is not a closed homomorphism

are given by the classical intersection theorem.

Theorem 2.1. If G is a grammar, $R \subseteq T(G)^*$ a regular set, then there exist a grammar G_R and a very fine homomorphism $\phi_R: G_R \to G$ with

i) $\phi_R(P(G_R)) = P(G)$,

ii) ϕ_R is surjective on objects,

iii) ϕ_R is the identity on terminals,

iv) $\mathscr{L}(G_R) = \mathscr{L}(G) \cap R$.

Proof. We use a slightly modified version of the classical construction of G_R [2]. Consider a finite state acceptor $a = (Q, X, \delta, F, q_0)$ with $T(a) = R$.

Now define G_R by

1) $Z(G_R) = Q \times Z(G) \times Q$,
$T(G_R) = T(G)$,
$s(G_R) = \{q_0\} \times s(G) \times F$.

2) $P(G_R)$ is constituted by the following productions. Let $\eta_s \ldots \eta_\cdot \to t_0 \xi_1 \ldots \xi_r t_r \in P(G)$ with

$$s > 0, \quad r \geqq 0, \quad \eta_i \in Z(G)\ (1 \leqq i \leqq s), \quad \xi_j \in Z(G),$$
$$(1 \leqq j \leqq r), \quad t_j \in T(G)^*\ (0 \leqq j \leqq r).$$

Then

$$(p_1, \eta_1, p_1') \ldots (p_r, \eta_s, p_s') \to t_0 (q_1, \xi_1, q_1')\, t_1 \ldots (q_r, \xi_r, q_r')\, t_r \in P(G_1)$$

where $r \geqq 0$, $p_i, p_i', q_i, q_i' \in Q$ and

$$q_1 = \delta(t_0, p_1)\ \&\ q_{i+1} = \delta(t_i, q_i')\ (1 \leqq i \leqq r)\ \&\ p_s' = \delta(t_r, q_r')$$

if $r = 0$ and

$$p_s' = \delta(t_0, p_1)$$

if $r = 0$.

The homomorphism ϕ_R is then given by the admissible monoidhomomorphism $h\colon (Z(G_R), T(G), s(G_R)) \to (Z(G), T(G), s(G))$ defined by

$$h((p, \xi, q)) = \xi \ ((p, \xi, q) \in Z(G_R))$$

and

$$h(t) = t \qquad (t \in T(G_R)).$$

Condition iii) gives rise to another definition. Consider a homomorphism $\phi\colon G_1 \to G_2$ and a subset $A \subseteq A(G_1)$. We call ϕ an A-morphism iff $\phi(\xi) = \xi$, if $\xi \notin A$. Of special interest are Z-morphisms ($A = Z(G_1)$!) and T-morphisms ($A = T(G_1)$!).

We call ϕ an *isomorphism* if ϕ is both on objects and morphisms bijective, ($G_1 \underset{\phi}{\hat{=}} G_2$ or $G_1 \hat{=} G_2$).

Note that an isomorphism is always very fine and closed.

We are now dealing with "subgrammars". Consider two grammars G_1 and G. G_1 is a *subgrammar* of G ($G_1 \subseteq G$) if

i) $Z(G_1) \subseteq Z(G)$, $s(G_1) \subseteq s(G)$, $T(G_1) \subseteq T(G)$ and $P(G_1) \subseteq P(G)$,

ii) $Z(G_1) \cap s(G) = s(G_1)$.

If G_1 and G_2 are subgrammars, then the "union" $G_1 \cup G_2$ and the "intersection" $G_1 \cap G_2$ are defined in a natural way. More difficult to deal with is the concept of *image* and *coimage*.

If $\phi\colon G_1 \to G_2$ is a homomorphism. $G_1' \subseteq G_1$ and $G_2' \subseteq G_2$ then

$$\phi(G_1') = \bigcap \{\tilde{G}_2 \subseteq G_2 \mid \phi((A(G_1')^*, A(\tilde{G}_1')^*)_{G_1'}) \subseteq (A(\tilde{G}_2)^*, A(\tilde{G}_2)^*)_{\tilde{G}_2}\}$$

is the *image* of G_1 *under* ϕ and

$$\phi^{-1}(G_2') = \bigcup \{\tilde{G}_1 \subseteq G_1 \mid \phi((A(\tilde{G}_1)^*, A(\tilde{G}_1)^*)_{\tilde{G}_1}) \subseteq (A(G_2')^*, A(G_2')^*)_{G_2'})$$

is the *coimage* of G_2' under ϕ.

Consider a fine homomorphism $\phi\colon G_1 \to G_2$. Then an admissible monoidhomomorphism h_ϕ is defined by $h_\phi(\xi) = \phi\xi$ ($\xi \in A(G_1)$). We get $G_{h_\phi} \subseteq G_2$ and a factorizationdiagram

$$\begin{tikzcd} G_1 \arrow[r, "\phi_{h_\phi}"] \arrow[dr, "\phi"'] & G_{h_\phi} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \\ & G_2 \end{tikzcd}$$

In the case h_ϕ is ε-free, we get

$$G_1 \subseteq G^{h_\phi}$$

and a factorizationdiagram

$$\begin{tikzcd} G_1 \arrow[dr, "\phi"'] & \subseteq & G_2^{h_\phi} \arrow[dl, phantom] \\ & & \end{tikzcd}$$

$$\begin{tikzcd} G_1 \arrow[r, phantom, "\subseteq"] \arrow[dr, "\phi"'] & G_2^{h_\phi} \arrow[d, "\phi^{h_\phi}"] \\ & G_2 \end{tikzcd}$$

Related to the concept of subgrammars is the concept of direct sums. [12]

Consider two grammars G_1 and G_2.

If $G_1 \cap G_2 = (\phi, \phi, \phi, \phi)$, we define $G_1 \oplus G_2 = G_1 \cup G_2$. With the inclusions ν_i $(i=1, 2)$ we get a sum-diagram

$$G_1 \xrightarrow{\nu_1} G_1 \oplus G_2 \xleftarrow{\nu_2} G_2.$$

Observe that ν_i are very fine ϕ-morphisms for $i=1.2$.

Another interesting sum-diagram is obtained in the case $G_1 \cap G_2 = (T, \phi, T, \phi)$, where T is a fixed alphabet. We define $G_1 + G_2 = G_1 \cup G_2$. With the inclusions ν_i $(i=1, 2)$ we get a sum-diagram relative to Z-morphisms.

$$G_1 \xrightarrow{\nu_1} G_1 + G_2 \xleftarrow{\nu_2} G_2.$$

Observe that ν_i are very fine ϕ-morphisms.

Remark.

i) $\mathscr{L}(G_1 + G_2) = \mathscr{L}(G_1) \cup \mathscr{L}(G_2)$,

ii) $\mathscr{L}(G_1 \oplus G_2) = \mathscr{L}(G_1) \cup \mathscr{L}(G_2)$.

Consider the following situation.

$$\begin{tikzcd} G_1 \arrow[r, "\nu_1"] \arrow[d, "\phi_1"'] & G_1 \oplus G_2 & G_2 \arrow[l, "\nu_2"'] \arrow[d, "\phi_2"] \\ G_1' \arrow[r, "\nu_1'"] & G_1' \oplus G_2' & G_2 \arrow[l, "\nu_2'"'] \end{tikzcd}$$

then there is a unique homomorphism

$$\phi_1 \oplus \phi_2 \colon G_1 \oplus G_2 \to G_1' \oplus G_2'$$

with

$$(\phi_1 \oplus \phi_2)\, \nu_i = \nu_i' \circ \phi_i \qquad (i=1,2)$$

Observe that properties of ϕ_i transfer to $\phi_1 \oplus \phi_2$, for example if ϕ_i are closed, then $\phi_1 \oplus \phi_2$ is closed.

We now turn our interest to products and factorizations.

Lemma 2.5. If $\phi: G_1 \to G_2$ is a fine homomorphism, then there is a commuting diagram

$$\begin{tikzcd} & G_3 \arrow[dr, "\phi_2"] & \\ G_1 \arrow[ur, "\phi_1"] \arrow[rr, "\phi"'] & & G_2 \end{tikzcd}$$

such that ϕ_1 is a fine T-morphism and ϕ_2 is a very fine Z-morphism.

Proof. Consider the admissible monoidhomomorphism, defined by

$$h(\xi) = \xi \quad (\xi \in Z(G_1)),$$
$$h(t) = \phi(t) \quad (t \in T(G_1)),$$
$$G_h = G_3 \quad \text{and} \quad \phi_1 = \phi_h.$$

We show

$$\text{``}\phi_1(r) = \phi_1(r') \Rightarrow \phi(r) = \phi(r')\text{''}$$

Let $r = (p, t_0\eta_1 \dots \eta_k t_k)$ and $r' = (p', t'_0\eta' \dots \eta'_l t'_l)$, where $t_i, t'_i \in T(G_1)^*$, η_i, η'_i, $p, p' \in Z(G_1)$.

Application of ϕ_1 yields

$$l = k, \quad p = p', \quad \eta_i = \eta'_i \,(1 \leqq i \leqq k) \quad \text{and} \quad \phi(t_i) = \phi(t'_i) \;(0 \leqq i \leqq k).$$

By this we get $\phi(r) = \phi(r')$.

With the above assertion the existence of ϕ_2 follows immediately.

Corollary. If ϕ is closed, then ϕ_2 is closed.

Theorem 2.2. To any pair $\phi_i: G_i \to G_0$ $(i = 1,2)$ of homomorphisms, where ϕ_1 is fine, ϕ_2 is very fine there exists a commuting diagram

$$\begin{tikzcd} G_0 \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \arrow[d, "\phi_3"] \\ G_2 \arrow[r, "\phi_4"'] & G_3 \end{tikzcd}$$

where ϕ_4 is fine and ϕ_3 is very fine.

Furthermore

i) If ϕ_1 is epic, then ϕ_4 is epic.
ii) If ϕ_1 is a T-morphism, then ϕ_4 is a T-morphism.
iii) If ϕ_2 is epic, then ϕ_3 is epic.

Proof. The proof is in its essence a pullback-construction ([10, 13]) in three steps.

Step 1. Assume "ϕ_2 is a Z-morphism". Define G_3 by

1) $Z(G_3) = \{(\xi, \xi') \in Z(G_1) \times Z(G_2) \mid \phi_1(\xi) = \phi_2(\xi')\}$, $T(G_3) = T(G_1)$,
$s(G_3) = \{(\sigma, \sigma') \in s(G_1) \times s(G_2) \mid \phi_1(\sigma) = \phi_2(\sigma')\}$.

2) $P(G_3)=\{((\eta_1, \eta_1') \ldots (\eta_s, \eta_s'), t_0(\xi_1, \xi_1') \ldots (\xi_r, \xi_r') t_r) \mid$
$s \geqq 0, \quad r \geqq 0, \quad t_i \in T(G_1)^* \; (0 \leqq i \leqq r),$
$(\eta_1 \ldots \eta_s, t_0\xi_1 \ldots \xi_r t_r) \in P(G_1)$ &
$(\phi_1(\eta_1 \ldots \eta_s), \phi_1(t_0\xi_1 \ldots \xi_r t_r)) \in \phi_2(P(G_2))\}.$

Now consider the monoidhomomorphisms h_3 and h_4 defined by

$$h_3(\xi, \xi')=\xi, \quad h_4(\xi, \xi')=\xi',$$
$$h_3(t)=t, \quad h_4(t)=\phi_1(t) \quad (t \in T(G_1)).$$

These two monoidhomomorphisms induce in a natural way homomorphisms ϕ_3 and ϕ_4 of the desired kind.

Step 2. Assume "ϕ_2 is a T-morphism". Define G_3 by

1) $Z(G_3)=Z(G_1), \quad T(G_1)=\{(t, w) \mid t \in T(G_1), w \in T(G_2)^*, \phi_1(t)=\phi_2(w)\},$
$s(G_3)=s(G_1),$

2) $P(G_3)=\{(p, \eta_0(t_1, w_1) \ldots \eta_{s-1}(t_s, w_s) \eta_s) \mid$
$s \geqq 0, \quad (t_i, w_i) \in T(G_3), \quad \eta_i \in Z(G_3)^*$ &
$(p, \eta_0 t_1 \ldots \eta_{s-1} t_s \eta_s) \in P(G_1)$ &
$(\phi_1(p), \phi_1(\eta_0) \phi_2(w_1) \ldots \phi_1(\eta_{s-1}) \phi_2(w_s) \phi_1(\eta_s)) \in \phi_2(P(G_2))\}.$

Again, define monoidhomomorphisms h_3 and h_4 by

$$h_3(\xi)=\xi, \quad h_4(\xi)=\phi_1(\xi) \quad (\xi \in Z(G_3)),$$
$$h_3(t, w)=t, \quad h_4(t, w)=w \quad ((t, w) \in T(G_3)).$$

In a natural way homomorphisms ϕ_3 and ϕ_4 of the desired kind are induced.

Step 3. We combine Step 1 and Step 2 considering the given diagram

$$\begin{tikzcd} G_1 \arrow[d, "\phi_1"'] & \\ G_0 & G_2 \arrow[l, "\phi_2"] \end{tikzcd}$$

By the factorization-lemma 2.5 we get the diagram

$$\begin{tikzcd} G_1 \arrow[dd, "\phi_1"'] & & \\ & G_0' \arrow[dl, "\psi_1"'] & \\ G_0 & & G_2 \arrow[ul, "\psi_2"'] \arrow[ll, "\phi_2"] \end{tikzcd}$$

where ψ_1 is a Z-morphism and ψ_2 is a T-morphism.

By Step 1 we get a diagram

$$\begin{tikzcd} G_1 \arrow[dd, "\phi_1"'] & G_3' \arrow[l, "\chi_2"'] \arrow[d, "\chi_1"] & \\ & G_0' \arrow[dl, "\psi_1"'] & \\ G_0 & & G_2 \arrow[ul, "\psi_2"'] \arrow[ll, "\phi_2"] \end{tikzcd}$$

By Step 2 we get a diagram

$$\begin{tikzcd}
G_1 \arrow[dd, "\phi_1"'] & G_3' \arrow[l, "\chi_2"'] \arrow[d, "\chi_1"] & G_3 \arrow[l, "\chi'_2"'] \arrow[dd, "\chi_1'"] \\
 & G_0' \arrow[dl, "\psi_2"'] & \\
G_0 & & G_2 \arrow[ll, "\phi_2"] \arrow[ul, "\psi_1"']
\end{tikzcd}$$

Now the statement of our theorem follows immediately.

3. Grammar Forms

Consider a grammar G. Following Ginsburg et al. [3], we define an *interpretation* μ of G to be a substitution $\mu\colon A(G)^* \to 2^{A(G')^*}$ where G' is another grammar such that the following conditions hold:

i) $\xi \in Z(G) \Rightarrow \mu(\xi) \subseteq Z(G')$,

ii) $s(G') \subseteq \mu(s(G))$,

iii) $\xi \neq \xi' \in Z(G) \Rightarrow \mu(\xi) \cap \mu(\xi') = \phi$,

iv) $t \in T(G) \Rightarrow \mu(t) \subseteq T(G')^*$ & $\mu(t)$ is finite,

v) $P(G') \subseteq \{(p', q') \mid (p, q) \in P(G): p' \in \mu(p)$ and $q' \in \mu(q)\}$.

We use the notion $\mu\colon G \Rightarrow G'$ for interpretations.

Theorem 3.1. The following statements are equivalent.

i) There exists $\mu\colon G \Rightarrow G'$.

ii) There exists a diagram

$$G \xleftarrow{\phi_1} G'' \xrightarrow{\phi_2} G'$$

where ϕ_1 is very fine and ϕ_2 is an epic and fine T-morphism.

Proof. i) $\Rightarrow$ ii): Consider $T = \{(t, t') \mid t' \in \mu(t),\ t \in T(G)\}$.
Define $h\colon (T \cup Z(G')^*) \to A(G)^*$ by

$$h(t, t') = t \quad \text{and} \quad h(\xi) = \eta \qquad (\xi \in Z(G'),\ \xi \in \mu(\eta)).$$

Obviously h is admissible and well-defined. Furthermore define $h_1\colon (T \cup Z(G'))^* \to A(G')^*$ by

$$h_1(t, t') = t' \;\&\; h_1(\xi) = \xi \qquad (\xi \in Z(G')).$$

Now consider ϕ_{h_1} and $(G^h)_{h_1}$.

By this we get the diagram

$$G \xleftarrow{\phi^h} G^h \xrightarrow{\phi_{h_1}} (G^h)_{h_1} \supseteq G'.$$

Then there exists a diagram

$$\begin{tikzcd} G & G^h \arrow[l, "\phi^h"'] \arrow[r, "\phi_{h_1}"] \arrow[d, phantom, "\cup"] & (G^h)_{h_1} \arrow[d, phantom, "\cup"] \\ \phi_{h_1}^{-1}(G')=G'' \arrow[rr, "\phi_2"'] & & G' \end{tikzcd}$$

which gives the desired diagram of ii).

ii) $\Rightarrow$ i): Consider the diagram

$$G \xleftarrow{\phi_1} G'' \xrightarrow{\phi_2} G'.$$

Since ϕ_1 is very fine there exists a length-preserving and admissible h such that

$$\begin{tikzcd} & (G^h) \arrow[dl, "\phi h"'] \arrow[d, phantom, "\cup"] \\ G & G'' \arrow[l, "\phi_1"] \end{tikzcd}$$

commutes.

Furthermore there exists an admissible h_1' such that the diagram

$$\begin{tikzcd} & (G'')_{h_1'} \arrow[dl, phantom, "\supseteq"] \\ G' & G'' \arrow[l, "\phi_2"] \arrow[u, "\phi_{h_1'}"'] \end{tikzcd}$$

commutes.

Hence we have reached the following situation

$$\begin{tikzcd} (G)^h \arrow[r] \arrow[d, phantom, "\cup"] & ((G)^h)_{h_1'} \arrow[d, phantom, "\cup"] \\ G'' \arrow[r] \arrow[dr] & (G'')_{h_1'} \arrow[d, phantom, "\cup"] \\ & G' \end{tikzcd}$$

A diagram

$$G \xleftarrow{\phi^h} (G)_h \xrightarrow{\phi_{h_1'}} (G^h)_{h_1'} \supseteq G'$$

results with $h_1'/Z((G)^h)$ is the identity.

Now define the interpretation μ by

$$\mu(\eta)=h^{-1}(\eta) \qquad (\eta \in Z(G))$$

and

$$\mu(t)=h_1' h^{-1}(t) \qquad (t \in T(G)).$$

We are now in the position to give our definition of grammar forms.

Definition 3.1. A *grammarform* Γ is a triple $\Gamma=(G, H_1, H_2)$, where G is a grammar and H_1, H_2 are classes of homomorphisms.

If Γ is a grammar form we associate a grammar family $|\Gamma|$ by defining

$$|\Gamma|=\{G_1 \mid \text{there exists a diagram } G \xleftarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_1 \text{ with } \phi_1 \in H_1,\ \phi_2 \in H_2\}$$

If $\mathcal{G}$ is a grammar family then

$$\mathcal{L}(\mathcal{G})=\{L \mid \exists G \in \mathcal{G}: \mathcal{L}(G)=L\}$$

is the associated language family.

In case $\mathcal{G}=|\Gamma|$ we write $\mathcal{L}(\mathcal{G})=\mathcal{L}(\Gamma)$.

Definition 3.2. A grammarform $\Gamma=(G, H_1, H_2)$ is *regular* if and only if

i) ϕ isomorphism $\Rightarrow \phi \in H_1 \cap H_2$,

ii) $\phi_1, \phi_2 \in H_1 \quad \Rightarrow \phi_1 \circ \phi_2 \in H_1$,

iii) $\phi_1, \phi_2 \in H_2 \quad \Rightarrow \phi_1 \circ \phi_2 \in H_2$,

iv) To any pair $\phi_i: G_i \to G_0$ $(i=1, 2)$ with $\phi_1 \in H_2$ and $\phi_2 \in H_1$ there exists a commuting diagram

$$\begin{tikzcd}
G_1 \arrow[d, "\phi_1"'] & G_3 \arrow[l, "\phi_3"'] \arrow[d, "\phi_4"] \\
G_0 & G_2 \arrow[l, "\phi_2"]
\end{tikzcd}$$

with $\phi_3 \in H_1$ and $\phi_4 \in H_2$.

The most significant axiom is "iv)". Theorem 2.2 is an example for this axiom.

Lemma 3.1. If $\Gamma=(G, H_1, H_2)$ is a regular grammar form, $G' \in |\Gamma|$, then $\Gamma'=(G', H_1, H_2)$ is regular and $|\Gamma'| \subseteq |\Gamma|$.

Proof. We have to show the inclusion. Consider a fixed diagram

$$G \xleftarrow{\phi_1} G'' \xrightarrow{\phi_2} G', \qquad \phi_1 \in H_1 \ \&\ \phi_2 \in H_2$$

for G' and an arbitrary diagram

$$G' \xleftarrow{\phi_1'} G_1' \xrightarrow{\phi_2'} G_1, \qquad \phi_1' \in H_1, \qquad \phi_2' \in H_2.$$

By axiom iv) there exists a diagram

$$\begin{tikzcd}
G & G'' \arrow[l, "\phi_1"'] \arrow[d, "\phi_2"] & G_3' \arrow[l, "\phi_3"'] \arrow[d, "\phi_4"] & \\
 & G' & G_1' \arrow[l, "\phi_1'"] \arrow[r, "\phi_2''"] & G_1
\end{tikzcd}$$

where $\phi_3 \in H_1$ and $\phi_4 \in H_2$.

Now by the axioms ii) and iii) we obtain $G_1 \in |\Gamma|$, and this proves the lemma.

Corollary. If $\Gamma=(G, H_1, H_2)$ and $\Gamma'=(G', H_1, H_2)$ are regular grammar forms then:

$$|\Gamma|=|\Gamma'| \Leftrightarrow G' \in |\Gamma| \ \&\ G \in |\Gamma'|.$$

We call a homomorphism $\phi: G_1 \to G_2$ ε-free if $\phi(w) \neq \varepsilon$ for all $w \in T(G_1)$, $w \neq \varepsilon$.

Theorem 3.2. Let $\Gamma=(G, H_1, H_2)$ be a regular grammar form.

1) If all very fine and epic Z-morphisms ϕ are in H_1, then $\mathscr{L}(\Gamma)$ is closed under intersection with regular sets.

2) If all fine and ε-free and epic T-morphisms are in H_2, then $\mathscr{L}(\Gamma)$ is closed under ε-free homomorphisms.

3) If all very fine and epic T-morphisms are in H_2 and all very fine and epic homomorphisms are in H_1, then $\mathscr{L}(\Gamma)$ is closed under union if H_1 and H_2 are closed under the operation '$\oplus$'.

Proof. 1) Consider $L=\mathscr{L}(G')$, $G'\in|\Gamma|$ and R regular, then there exists a diagram

$$G \xleftarrow{\phi_1} G'' \xrightarrow{\phi_2} G' \xleftarrow{\psi_1} G'_R$$

where $\phi_1\in H_1$, $\phi_2\in H_2$ and $\psi_1\in H_1$.

By axiom iv) we can "fill-in" this diagram and get

$$\begin{tikzcd} G & G'' \arrow[l, "\phi_1\in H_1"'] \arrow[d, "\phi_2\in H_2"'] & G''_1 \arrow[l, "\phi_3\in H_1"'] \arrow[d, "\phi_4\in H_2"] \\ & G' & G'_R \arrow[l, "\psi_1\in H_1"] \end{tikzcd}$$

Now $\phi_1\phi_3\in H_1$ and 1) is following.

2) Consider an ε-free homomorphism h and $G'\in|\Gamma|$, then we get a diagram

$$G \xleftarrow{\phi_1\in H_1} G'' \xrightarrow{\phi_2\in H_2} G' \xrightarrow{\phi_h} G'_h.$$

Since $\phi_h\in H_2$ we get the assertion by observing $\phi_h\phi_2\in H_2$.

3) First we observe the following fact
"If G is a grammar, $T\supseteq T(G)$ an alphabet, then there exists a grammar G' with $T(G')=T$, $\mathscr{L}(G')=\mathscr{L}(G)$ and a very fine and epic T-morphism $\phi\colon G'\to G''$.

This fact means, we can always add unnecessary terminals. Now, consider two arbitrary diagrams

$$G \xleftarrow{\phi_1\in H_1} G'_1 \xrightarrow{\phi_2\in H_2} G_1 \ \&$$

$$G \xleftarrow{\phi'_1\in H_1} G'_2 \xrightarrow{\phi'_2\in H_2} G_2.$$

By our observation we can assume $T(G_1)=T(G_2)=T$. By axiom i) we can find

$$\tilde{G} \underset{\tilde{\phi}_0}{\triangleq} G, \quad \tilde{G}'_1 \underset{\tilde{\phi}_1}{\triangleq} G'_1, \quad \tilde{G} \underset{\tilde{\phi}_2}{\triangleq} G_1, \quad \hat{G}_1 \underset{\tilde{\phi}_3}{\triangleq} G_1,$$

such that $T(\hat{G}_1)=T(G_1)$ and

$$G\oplus\tilde{G}, \quad \tilde{G}'_1\oplus G'_2, \quad \tilde{G}_1\oplus G_2, \quad \tilde{G}_1+G_2$$

are defined, and $\tilde{\phi}_i$ $(i=0, 1, 2, 3)$ are all very fine and epic T-morphisms.

Moreover we can find a very fine and epic T-morphism $\chi': \tilde{G}_1 \oplus G_2 \to \hat{G}_1 + G_2$ and a very fine and epic homomorphism $\chi: \tilde{G} \oplus G \to G$.

Then we get the following diagram

$$\begin{tikzcd}
 & \tilde{G} \arrow[d] & \tilde{G}'_1 \arrow[l, "\tilde{\phi}_1"'] \arrow[r, "\tilde{\phi}_2"] \arrow[d] & \tilde{G}_1 \arrow[d] & \\
G & \tilde{G} \oplus G \arrow[l, "\chi"'] & \tilde{G}'_1 \oplus G'_2 \arrow[l, "\tilde{\phi}_1 \oplus \phi'_1"'] \arrow[r, "\tilde{\phi}_2 \oplus \phi'_2"] & \tilde{G}_1 \oplus G_2 \arrow[r, "\chi'"] & \hat{G}_1 + G_2 \\
 & G \arrow[u] & G'_2 \arrow[l, "\phi'_1"] \arrow[r, "\phi'_2"'] \arrow[u] & G_2 \arrow[u] &
\end{tikzcd}$$

where $\tilde{\phi}_1$, $\phi'_1 \in H_1$ (axioms i) and ii)) and $\tilde{\phi}_2$, $\phi'_2 \in H_2$ (axioms i) and iii)).

Now our assumption on the operation '$\oplus$' and the axioms ii) and iii) give the desired result.

One of the main point of our proofs is the possibility of dualization by changing the direction of arrows. If $\Gamma = (G, H_1, H_2)$ is a grammar form, then the *cofamily* $|\Gamma|^{co}$ is defined by

$$|\Gamma|^{co} = \{G_1 \mid \text{There exists a diagram } G \xrightarrow{\phi_1} G_2 \xleftarrow{\phi_2} G_1 \text{ where } \phi_1 \in H_1 \text{ and } \phi_1 \in H_2\}.$$

Analogously, $\mathscr{L}^{co}(\Gamma) = \mathscr{L}(|\Gamma|^{co})$ is the associated language *cofamily*.

In dualizing we define "*coregular*" be leaving axioms i), ii) and iii) unchanged, and dualizing axiom iv) in the following way axiom iv)^{co}: To any pair $\phi_i: G_0 \to G_i$ $(i=1, 2)$ with $\phi_1 \in H_2$ and $\phi_2 \in H_1$ there exists a commuting diagram

$$\begin{tikzcd}
G_1 \arrow[d, "\phi_1"'] & G_3 \arrow[l, "\phi_3"'] \arrow[d, "\phi_4"] \\
G_0 & G_2 \arrow[l, "\phi_2"]
\end{tikzcd}$$

with $\epsilon_3 \in H_1$ and $\phi_4 \in H_2$.

Remark. Examples for axiom iv)^{co} can be given by using congruence lattices [10].

By dualization we get the following lemmas and theorems, where the duality is denoted by the superscript "co".

We indicate by two proof-examples how dualization should work.

Lemma 3.1^{co}. If $\Gamma = (G, H_1, H_2)$ is a coregular grammar form, $G' \in |\Gamma|^{co}$ then $\Gamma' = (G', H_1, H_2)$ is coregular and $|\Gamma'|^{co} \subseteq |\Gamma|^{co}$.

Proof. Consider a fixed diagram

$$G \xrightarrow{\phi_1} G'' \xleftarrow{\phi_2} G' \quad \text{with } \phi_1 \in H_1 \;\&\; \phi_2 \in H_2$$

and an arbitrary diagram

$$G' \xrightarrow{\phi_1'} G_1' \xleftarrow{\phi_2'} G_1 \quad \text{with } \phi_1' \in H_1, \quad \phi_2' \in H_2.$$

By axiom ${}^{\text{co}}$iv there exists a diagram

$$\begin{tikzcd} G \arrow[r, "\phi_1"] & G'' \arrow[d, "H_1 \in \phi_3"'] & G' \arrow[l, "\phi_2"'] \arrow[d, "\phi_1'"] & \\ & G_4 & G_1' \arrow[l, "\phi_4 \in H_2"] & G_1. \arrow[l, "\phi_2'"'] \end{tikzcd}$$

Now by axiom ii) and iii) we get $G_1 \in |\Gamma|^{\text{co}}$, which proves the lemma.

Corollary$^{\text{co}}$. If $\Gamma=(G, H_1, H_2)$ and $\Gamma'=(G', H_1, H_2)$ are coregular grammar forms then

$$|\Gamma|^{\text{co}}=|\Gamma'|^{\text{co}} \Leftrightarrow G' \in |\Gamma|^{\text{co}} \;\&\; G \in |\Gamma'|^{\text{co}}.$$

Theorem 3.2$^{\text{co}}$. Let $\Gamma=(G, H_1, H_2)$ be a coregular grammar form.

1) If all very fine and epic Z-morphisms are in H_2, then $\mathscr{L}^{\text{co}}(\Gamma)$ is closed under intersection with regular sets.

2) If all fine and ε-free T-morphisms are in H_1, then $\mathscr{L}^{\text{co}}(\Gamma)$ is closed under ε-free homomorphisms.

3) If all very fine and epic T-morphisms are in H_1 and all very fine and epic homomorphisms are in H_1, then $\mathscr{L}^{\text{co}}(\Gamma)$ is closed under union provided H_1 and H_2 are closed under the operation '$\oplus$'.

Proof. 1) Follows immediately by axiom iii).

2) Consider a diagram

$$G \xrightarrow{\phi_1} G'' \xleftarrow{\phi_2} G' \xrightarrow{\phi_0} G_1$$

with $\phi_1 \in H_1$, $\phi_2 \in H_2$, ϕ_0 fine and ε-free T-morphism. Since $\phi_0 \in H_1$ we get by axiom ${}^{\text{co}}$iv) a diagram

$$\begin{tikzcd} G \arrow[r, "\phi_1"] & G'' \arrow[d, "\phi_0'"'] & G' \arrow[l, "\phi_2"'] \arrow[d, "\phi_0"] \\ & G_1' & G_1 \arrow[l, "\phi_2'"] \end{tikzcd}$$

where $\phi_0' \in H_1$, $\phi_2' \in H_2$. Now by axiom ii) we get $G_1 \in |\Gamma|^{\text{co}}$.

3) Dualize part 3) of the proof of Theorem 3.2, where the sum-diagram is considered to be self-dual.

Notation. If $\Gamma=(G, H_1, H_2)$, then $\Gamma^{\text{co}}=(G, H_2, H_1)$.

Lemma 3.2. If Γ is regular then

$$|\Gamma^{\text{co}}|^{\text{co}} \subseteq |\Gamma|.$$

Proof. Consider a diagram $G \xrightarrow{\phi_1 \in H_2} G'' \xleftarrow{\phi_2 \in H_1} G'$, then by axiom iv) we can fill in this diagram by

$$\begin{array}{ccc} G & \xleftarrow{\phi_3 \in H_1} & G''' \\ {\scriptstyle \phi_1 \in H_2}\big\downarrow & & \big\downarrow{\scriptstyle \phi_4 \in H_2} \\ G'' & \xleftarrow[\phi_2 \in H_1]{} & G' \end{array}$$

But this proves $G' \in |\Gamma|$.

Dualizing yields

Lemma 3.2^{co}. If Γ is coregular, then

$$|\Gamma^{co}| \subseteq |\Gamma|^{co}.$$

Corollary. If Γ is regular and Γ^{co} coregular, then

$$|\Gamma| = |\Gamma^{co}|^{co}.$$

Proof. Observe

$$|\Gamma| = |(\Gamma^{co})^{co}| \subseteq |\Gamma^{co}|^{co}.$$

We now turn our interest to grammar indices which have been introduced by A. Salomaa [9].

Consider a grammar G and a derivation f. Let $\Delta(f)$ be the set of all sequential representations of f. To any $\delta \in \Delta(f)$,

$$\delta \equiv (1_{u_1} \times r_1 \times 1_{v_1}) \circ \cdots \circ (1_{u_s} \times r_s \times 1_{v_s})$$

we assign

$$\#(\delta) = \operatorname{Max}\{|u_i d(r_i) v_i|_Z \mid 1 \leqq i \leqq s\}$$

where $|w|_Z$ is the number of variables in w.

Then we can assign to f a number

$$\#(f) = \operatorname{Min}\{\#(\delta) \mid \delta \in \Delta(f)\}$$

and define the *index* of G to be

$$\operatorname{index}(G) = \operatorname{Sup}\{\#(f) \mid f \in D(G)\}$$

We study the behaviour of the index function under homomorphisms.

Lemma 3.3. i) $G_1 \subseteq G_2 \Rightarrow \operatorname{index}(G_1) \leqq \operatorname{index}(G_2)$.

ii) If $\phi: G_1 \to G_2$ is fine, then $\operatorname{index}(G_1) \leqq \operatorname{index}(G_2)$.

Proof. i) and ii) is trivial:

ii) We have to show:

$$\operatorname{index}(\phi(f)) = \operatorname{index}(f).$$

Since ϕ is on variables length preserving the above assertion follows from the fact, that ϕ induces a surjective mapping from $\Delta(f)$ onto $\Delta(\phi(f))$.

But this is true, since all sequential representations of f can be generated by applying successively the following rules to a fixed sequential representation

$$\text{“}(1_u \times r \times 1_{v_1 c(r') v_2}) \circ (1_{u d(r) v_1} \times r' \times 1_{v_2}) \equiv (1_{u c(r) v_1} \times r' \times 1_{v_2}) \circ (1_u \times r \times 1_{v_1 d(r') v_2})$$

and $$(1_{u_1 c(r') u_2} \times r \times 1_v) \circ (1_{u_1} \times r' \times 1_{u_2 d(r) v}) \equiv (1_{u_1} \times r' \times 1_{u_2 c(r) v}) \circ (1_{u_1 d(r') u_2} \times r \times 1_v)\text{”}$$

and these rules are transformed into the corresponding rules via ϕ.

Now we can prove a generalization of a theorem of [3].

Theorem 3.3. Consider a grammarform $\Gamma = (G, H_1, H_2)$ where

i) $\phi \in H_1 \Rightarrow \phi$ is fine,

ii) $\phi \in H_2 \Rightarrow \phi$ is closed and fine,

iii) index $(G) < \infty$,

then $\mathscr{L}(\Gamma)$ is contained in the class of nonexpansive languages [4].

Proof. By Lemma 3.3 we know

$$G' \in |\Gamma| \Rightarrow \text{index}\,(G') \leqq \text{index}\,(G) < \infty.$$

Now a theorem of A. Salomaa [9] states:

If G_1 is a grammar

$$\text{Sup Inf}\,\{\overline{\#}(\text{f}) \mid d(f) \in s(G_1\ \&\ c(f) = w\} < \infty \qquad w \in \mathscr{L}(G_1)$$

then $\mathscr{L}(G_1)$ is nonexpansive.

But this proves our theorem.

Again we have a dual version of the theorem.

Theorem 3.3$^{\text{co}}$. Consider a grammar form $\Gamma = (G, H_1, H_2)$ where

i) $\phi \in H_1 \Rightarrow \phi$ is closed and fine,

ii) $\phi \in H_2 \Rightarrow \phi$ is fine,

iii) index $(G) < \infty$,

then $\mathscr{L}^{\text{co}}(\Gamma)$ is contained in the class of nonexpansive languages.

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