INDEX SETS IN THE ARITHMETICAL HIERARCHY

Ulrike BRANDT


Communicated by D. van Dalen
Received 26 January 1985; revised 10 November 1986

We prove the following results: every recursively enumerable set approximated by finite sets of some set \( M \) of recursively enumerable sets with index set in \( \Pi_2 \) is an element of \( M \), provided that the finite sets in \( M \) are canonically enumerable. If both the finite sets in \( M \) and in \( M \) are canonically enumerable, then the index set of \( M \) is in \( \Sigma_2 \cap \Pi_2 \) if and only if \( M \) consists exactly of the sets approximated by finite sets of \( M \) and the complement \( \tilde{M} \) consists exactly of the sets approximated by finite sets of \( M \). Under the same condition \( M \) or \( \tilde{M} \) has a non-empty subset with recursively enumerable index set, if the index set of \( M \) is in \( \Sigma_2 \cap \Pi_2 \).

If the finite sets in \( M \) are canonically enumerable, then the following three statements are equivalent: (i) the index set of \( M \) is in \( \Sigma_2 \setminus \Pi_2 \), (ii) the index set of \( M \) is \( \Sigma_2 \)-complete, (iii) the index set of \( M \) is in \( \Sigma_2 \) and some sequence of finite sets in \( M \) approximate a set in \( \tilde{M} \).

Finally, for every \( n \geq 2 \), an index set in \( \Sigma_n \setminus \Pi_n \) is presented which is not \( \Sigma_n \)-complete.

Introduction

The well known theorems of Rice and Rice–Shapiro [9] characterize sets of indices (index sets) of sets of partial recursive (p.r.) functions located in \( \Sigma_0 \) and \( \Sigma_1 \). An index set is the set of all ‘programs’ computing the functions of the given set. Up to now there has been no such characterization for higher steps in the arithmetical hierarchy. The interest in studying index sets located on higher steps of the hierarchy—especially between \( \Sigma_3 \cap \Pi_3 \) and \( \Sigma_2 \cap \Pi_2 \)—is motivated by results related to the inductive inference problem. It can be shown that any identifiable function set is included in an identifiable function set with index set in \( \Sigma_3 \cap \Pi_3 \) [2], [3]. Thus, there is an obvious desire to get more informations about function sets with index sets in \( \Sigma_3 \cap \Pi_3 \). None of the identifiable sets can include the whole set of recursive functions (Gold [4]). Moreover with the help of Gold’s result and the Rice–Shapiro theorem it can be shown that no identifiable set can include a non-empty subset with index set in \( \Sigma_1 \); hence no non-empty function set with index set in \( \Sigma_1 \) is identifiable [3].

We want to present three results on this topic: every recursively enumerable (r.e.) set approximated by finite sets of some set \( M \) of r.e. sets with index set in \( \Pi_2 \) is an element of \( M \), provided that the finite sets in \( M \) are canonically enumerable. If both the finite sets in \( M \) and in \( \tilde{M} \) are canonically enumerable, then the index set of \( M \) is in \( \Sigma_2 \cap \Pi_2 \) if and only if \( M \) consists exactly of the sets.
approximated by finite sets of $M$ and the complement $\tilde{M}$ consists exactly of the sets approximated by finite sets of $\tilde{M}$. Under the same condition $M$ or $\tilde{M}$ has a non-empty subset with r.e. index set, if the index set of $M$ is in $\Sigma_2 \cap \Pi_2$.

Furthermore we investigate $\Sigma_2$-completeness of index sets. The relation between intuitive simplicity of definition and completeness or non-completeness of arithmetical sets is a problem not fully understood [9, p. 330]. Almost all index sets studied in the past have been proved to be $\Sigma_n$-complete or $\Pi_n$-complete. The question of completeness of index sets has been further discussed by D. E. Miller. He shows in [8] that every naturally defined class of sets includes an index set which is 1-complete for that class and that every index set is 1-complete for some naturally defined class, where 'naturally defined' is formalized as 'effective Boolean'.

According to Rice's theorem (or, more precisely, its proof as given in Rogers [9]) every non-trivial index set in $\Sigma_1$ is $\Sigma_1$-complete. Assuming that the finite sets in $M$ are canonically r.e., we shall show, that the following three statements are equivalent:

(i) The index set of $M$ is in $\Sigma_2 \setminus \Pi_2$.

(ii) The index set of $M$ is $\Sigma_2$-complete.

(iii) The index set of $M$ is in $\Sigma_2$ and some sequence of finite sets in $M$ approximate a set in $\tilde{M}$.

If we drop the assumption that the finite sets in $M$ are canonically r.e., then such a characterization seems to be rather difficult. This is indicated by the fact that there are index sets in $\Sigma_2 \setminus \Pi_2$ which are not complete on that level of the arithmetical hierarchy. Using the result of Yates [10] that $\{z \mid W_z =^TA\}$ is $\Sigma_4^A$-complete, Rogers [9] presents an index set in $\Sigma_4 \setminus \Pi_4$ which is not $\Sigma_4$-complete, choosing an appropriate $A$. This example is easily modified to provide for all $n > 1$ examples of index sets in $\Sigma_n \setminus \Pi_n$ which are not $\Sigma_n$-complete: let $A$ be the well-known example (due to Lachlan and Sacks, see e.g. Theorem 13-XXVI of [9]) of an r.e. set satisfying $\emptyset^{(n)} <_T A^{(n)} <_T \emptyset^{(n+1)}$ for all $n$. As remarked in Theorem 5.2 of [5] $\{i \mid W_i \cap B' \neq \emptyset\}$ is $\Sigma_1^B$-complete for all sets $B$; hence for all $n \geq 1$, $C_n = \{i \mid W_i \cap A^{(n)} \neq \emptyset\} =_m A^{(n)}$ is an example of an index set which is in $\Sigma_{n+1} \setminus \Pi_{n+1}$ but not $\Sigma_{n+1}$-complete.

In the last part of this paper we provide other examples of index sets on every level $>1$ of the arithmetical hierarchy which are not complete on that level. The examples are 'complementary' to the examples above in that they have the form $\{i \mid W_i \subseteq S\}$ while the above examples have form $\{i \mid W_i \notin S\}$.

**Basic notations and definitions**

We assume that the reader is familiar with the basic concepts and results of recursion theory [9]. We adopt the notations of [9]; in particular an acceptable enumeration of the unary p.r. functions is denoted by $(\phi_i)_{i=0}$. For any given $\phi_i$, $W_i$
is the domain of $\phi_i$ ($\text{dom}(\phi_i) = W_i$). $W_i^n$ is the set of all $x$ (with respect to a fixed dovetailing procedure) occurring within the first $n$ steps of the enumeration of $W_i$ (depending on the index $i$). The notation transfers to the relativized theory by indexing with the oracle $(\phi_i^B, W_i^B, W_i^{B,n})$. If $M$ is a set of r.e. sets, then $\text{Ind}(M) = \{i \mid \phi_i \in M\}$ is the index set of $M$.

The finite set with canonical index $x$ is $D_x$ [9]. Suppose a set $M$ of r.e. sets is given. The finite sets in $M$ are canonically enumerable (c.e.) if and only if there is a recursive function $f$ with $\{D_{f(x)} \mid x \in \mathbb{N}\} = \{W \mid W \in M \text{ and } W \text{ finite}\}$. $M$ is called canonical if and only if the finite sets in $M$ are canonically enumerable.

The next definition deals with the concept of approximating r.e. sets by finite sets. A r.e. set $A$ is called approximable by finite sets in $M$ if and only if for every $D_x \subseteq A$ there exists a $D_y \in M$ with $D_y \subseteq D_x \subseteq A$. By $\text{Approx}(M)$ we denote the set of all r.e. sets $A$ which are approximable by finite sets in $M$. According to the definition any finite set is in $M$ if and only if it is an element of $\text{Approx}(M)$.

Consider for example the set $M = \{J \mid \exists n \in \mathbb{N} : J = [0:n]\}$. Then there exists exactly one infinite set approximated by finite sets in $M$, namely the set $\mathbb{N}$. We get $\text{Approx}(M) = M \cup \{\mathbb{N}\}$.

In the following let $E$ denote the set of all finite subsets of $\mathbb{N}$.

**Index sets in $\Sigma_2$ and $\Sigma_2 \cap \Pi_2$**

We fix a set $M$ of r.e. sets. We want to show first that $\text{Approx}(M)$ is a subset of $M$, provided that $M$ is canonical and $\text{Ind}(M) \in \Pi_2$.

**Example.** Consider $M = \{A \mid A$ r.e. $\land ((x = \mu y : y \in A) \Rightarrow D_x \subseteq A)\}$. $M$ includes every nonempty r.e. set $A$ whose minimal element is a canonical index for a finite set included in $A$, therefore

$$M = \{A \mid A \text{ r.e. } \land \forall x, y [(x \in A \land (y < x \Rightarrow y \notin A)) \Rightarrow D_x \subseteq A]\}$$

$$= \{A \text{ r.e. } \mid \exists x (x \in A \land D_x \subseteq A \land \forall y (y \in A \Rightarrow y \geq x)) \lor \forall x (x \notin A)\}.$$

With the help of the Tarski–Kuratowski algorithm [9] we get

$$\text{Ind}(M)$$

$$= \{i \mid \forall x, y [(\exists n (x \in W_i^n \land (y < x \Rightarrow \forall m (y \notin W_i^m))) \Rightarrow \exists l (D_x \subseteq W_i^l)]\}$$

$$= \{i \mid \forall x, y, n \exists m, l [(x \in W_i^n \land (y < x \Rightarrow y \notin W_i^m)) \Rightarrow D_x \subseteq W_i^l]\}$$

as well as

$$\text{Ind}(M) = \{i \mid \exists x, n, l \forall y, m, k$$

$$[(x \in W_i^n \land D_x \subseteq W_i^l \land (y \in W_i^m \Rightarrow y \geq x)) \lor W_i^k = \emptyset]\}.$$

Hence $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$.

The finite sets in $M$ are c.e., even recursive since there are algorithms which
compute for any \( x \) the cardinality of \( D_x \) and the elements of \( D_x \) and decide for any \( x \) and \( y \) whether \( D_x \) is contained in \( D_y \) or not, hence the characteristic function

\[
\chi(x) = \begin{cases} 
1, & \text{if } (y = \mu z : z \in D_x) \Rightarrow D_y \subseteq D_x \\
0, & \text{otherwise}
\end{cases}
\]

is recursive. Therefore there exists recursive functions \( f \) and \( h \) with \( f(\mathbb{N}) = \chi^{-1}(1) \) and \( h(\mathbb{N}) = \chi^{-1}(0) \). By this

\[
\{ D_f(x) \mid x \in \mathbb{N} \} = M \cap E \quad \text{and} \quad \{ D_{h(x)} \mid x \in \mathbb{N} \} = \overline{M} \cap E,
\]

i.e., \( M \) and \( \overline{M} \) are both canonical.

We shall prove a lemma asserting that \( \operatorname{Approx}(M) \subseteq M \). Consider again the example and a set \( A \in \overline{M} \) with minimal element \( z \). By definition \( D_z \notin A \), i.e., \( D_z \notin D_y \) for every \( D_y \subseteq A \). Hence we can state for every \( D_y \subseteq A \) containing the element \( z \) that \( z \) is the minimal element in \( D_y \) and \( D_z \notin D_y \), which means \( D_y \in \overline{M} \). Thus \( A \notin \operatorname{Approx}(M) \).

**Lemma 1.** Every canonical set \( M \) has the following properties:

(i) If \( \operatorname{Approx}(M) \notin M \), then \( \operatorname{Ind}(E) \leq_m \operatorname{Ind}(M) \).

(ii) If \( M \notin \operatorname{Approx}(M) \), then \( \operatorname{Ind}(E) \leq_m \operatorname{Ind}(\overline{M}) \).

**Proof.** Consider a canonical set \( M \) and a recursive function \( f \) enumerating the finite sets of \( M \).

(i) If \( \operatorname{Approx}(M) \notin M \), there exists an \( A \in \operatorname{Approx}(M) \setminus M \) such that \( A \) is infinite since \( \operatorname{Approx}(M) \cap E \subseteq M \). Fix a recursive function \( g \) with \( g(\mathbb{N}) = A \). Define

\[
A^{(0)} = \{ g(0) \}, \quad A^{(n+1)} = A^{(n)} \cup \{ g(n+1) \}
\]

and

\[
p(0) = f(\mu y : \exists k (A^{(0)} \subseteq D_{f(y)} \subseteq A^{(k)})),
\]

\[
p(n+1) = f(\mu y : \exists k (A^{(n)} \cup D_{p(n)} \subseteq D_{f(y)} \subseteq A^{(k)})).
\]

Then \( p \) is recursive, \( D_{p(n)} \subseteq D_{p(n+1)} \subseteq A \) for all \( n \), and \( \forall n \ (x \in D_{p(n)}) \) iff \( x \in A \); hence \( \bigcup_n D_{p(n)} = A \). Define by the \( s_n^m \)-theorem a recursive function \( \beta(i) \) to be the index of an r.e. set defined by

\[
W_{\beta(i)} = \bigcup_{z \in y (z \in W_i)} D_{p(y)} = \{ x \mid \exists y \exists z \geq y (z \in W_i \land x \in D_{p(y)}) \}.
\]

If \( W_i \) is finite and \( m = \max \{ y \mid y \in W_i \} \), then \( W_{\beta(i)} = D_{p(m)} \in M \). If \( W_i \) is infinite, then \( W_{\beta(i)} = \bigcup_n D_{p(n)} = A \notin M \). Hence \( W_i \) is finite iff \( \beta(i) \in \operatorname{Ind}(M) \). In conclusion \( \operatorname{Ind}(E) \leq_m \operatorname{Ind}(M) \).

(ii) If \( M \notin \operatorname{Approx}(M) \) there exists an \( A \in M \setminus \operatorname{Approx}(M) \), i.e., there is an \( D_x \subseteq A \) such that \( \forall y (D_x \subseteq D_{f(y)} \Rightarrow D_y (z) \notin A) \). By this every finite subset of \( A \) properly including \( D_x \) is an element of \( \overline{M} \). As in the first part of the proof let \( A^{(0)} = \{ g(0) \}, \ A^{(n+1)} = A^{(n)} \cup \{ g(n+1) \} \). Define by the \( s_n^m \)-theorem a recursive
function $\beta(i)$ to be the index of an r.e. set defined by

$$W_{\beta(i)} = \bigcup_{n \in W_i} D_x \cup A^{(n)}.$$ 

If $W_i$ is infinite, $W_{\beta(i)} = A \in M$. If $W_i$ is finite, we get $D_x \subseteq W_{\beta(i)} \subseteq A$ where $W_{\beta(i)}$ is finite and hence $\in \tilde{M}$. In conclusion $\text{Ind}(E) \leq_m \tilde{M}$. □

As an immediate consequence of Lemma 1 we get

**Lemma 2.** Every canonical set $M$ has the following properties:

(i) If $\text{Ind}(M) \in \Pi_2$, then $\text{Approx}(M) \subseteq M$.

(ii) If $\text{Ind}(M) \in \Sigma_2$, then $M \subseteq \text{Approx}(M)$.

Look at the running example, for which we know $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$. We have shown $\text{Approx}(M) \subseteq M$. Now consider $A \in M$ with minimal element $z$ so that $D_z \subseteq A$. Now, any $D_y$ with $\{z\} \cup D_z \subseteq D_y \subseteq A$ has to be an element of $M$ since $z$ is its minimal element and $D_z \subseteq D_y$. To every $D_x \subseteq A$ such a $D_y$ containing $D_z$ can be found (namely $D_x \cup \{z\} \cup D_z$). Thus $A \in \text{Approx}(M)$. In conclusion we get $\text{Approx}(M) = M$.

We can now characterize the sets with $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$ as those sets with $\text{Approx}(M) = M$ and $\text{Approx}(\tilde{M}) = \tilde{M}$ provided that $M$ and $\tilde{M}$ are canonical.

**Lemma 3.** If $M$ is canonical and $M = \text{Approx}(M)$, then $\text{Ind}(M) \in \Pi_2$.

**Proof.** Choose a recursive function $f$ enumerating the canonical indices of the finite sets in $M$. If

$$M = \text{Approx}(M) = \{A \mid A \text{ r.e. } \land \forall D_x \subseteq A \exists D_y \in M : D_x \subseteq D_y \subseteq A\}$$

we get

$$M = \{A \mid \text{r.e. } \land \forall x \exists y (D_x \subseteq A \Rightarrow D_x \subseteq D_{f(y)} \subseteq A)\}.$$ 

Thus

$$\text{Ind}(M) = \{i \mid \forall x \exists y (D_x \subseteq W_i \Rightarrow D_x \subseteq D_{f(y)} \subseteq W_i)\}$$

$$= \{i \mid \forall x, n \exists y, m (D_x \subseteq W_i^n \Rightarrow D_x \subseteq D_{f(y)} \subseteq W_i^m)\} \in \Pi_2.$$ □

**Theorem 4.** If $M$ is a set of r.e. sets such that $M$ and $\tilde{M}$ are canonical, then $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$ if and only if $M = \text{Approx}(M)$ and $\tilde{M} = \text{Approx}(\tilde{M})$.

**Proof.** The only-if-part follows by Lemma 2, while the if-part is an immediate consequence of Lemma 3. □

If we consider canonical sets $M$ with the property $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$ and $\tilde{M}$ canonical, then we can show with the help of the Rice–Shapiro theorem that
there exists a set \( L \neq \emptyset \) such that \( L \subseteq M \) or \( L \subseteq \tilde{M} \), with \( \text{Ind}(L) \) r.e. In our running example we can choose

\[
L = \{ A \mid A \text{ r.e. } \land (0) \cup D_0 \subseteq A \}.
\]

According to the definition of \( M, \) \( L \subseteq M \). It is easily seen that \( \text{Ind}(L) \) is r.e., or equivalently \( \text{Ind}(L) \in \Sigma_1 \).

A set \( L \) of sets is called a basic open set if \( L = \{ A \text{ r.e. } \mid D_x \subseteq A \} \) for some \( D_x \) (see also p. 357 of [6]).

**Theorem 5.** If \( M \) and \( \tilde{M} \) are canonical and \( \text{Ind}(M) \in \Sigma_2 \cap \Pi_2 \), then \( M \) or \( \tilde{M} \) includes a non-empty basic open set.

**Proof.** Choose some recursive functions \( f \) and \( h \) enumerating the canonical indices of the finite sets in \( M \) and \( \tilde{M} \). Now we proceed by contradiction supposing that neither \( M \) nor \( \tilde{M} \) include a basic open set, i.e., for every \( x \), \( \{ A \text{ r.e. } \mid D_x \subseteq A \} \notin M \) and \( \{ A \text{ r.e. } \mid D_x \subseteq A \} \notin \tilde{M} \). Hence we can find for every \( D_x \in M \) a \( D_y \in \tilde{M} \) with \( D_x \neq D_y \). Otherwise there is an infinite r.e. \( B \in \tilde{M} \cap \text{Approx}(M) \) — a contradiction to Theorem 4. By the same argument there is for every \( D_y \in \tilde{M} \) a \( D_x \in M \) with \( D_y \neq D_x \). The fact that \( M \) and \( \tilde{M} \) are canonical even allows us to compute for every \( D_y \in \tilde{M} \) a \( D_x \in M \) with \( D_x \neq D_y \) using an algorithm computing \( y = \mu z (D_x \subseteq D_{f(z)}) \). The same holds for \( D_x \in M \): If \( x = \mu z (D_y \subseteq D_{h(z)}) \) we obtain \( D_y \neq D_x \). Now define a function \( p \) by

\[
\begin{align*}
  p(0) &= \mu z (z \in f(\mathbb{N})), \\
  p(2n+1) &= h(\mu z (D_{p(2n)} \subseteq D_{h(z)})), \\
  p(2n+2) &= f(\mu z (D_{p(2n+1)} \subseteq D_{f(z)})).
\end{align*}
\]

\( p \) is recursive. Define \( A = \bigcup_n D_{p(n)} \). Then \( A \) is an infinite r.e. set. By construction \( A \in \text{Approx}(M) \cap \text{Approx}(\tilde{M}) \) — a contradiction to Theorem 4.

For every basic open set \( L \), \( \text{Ind}(L) \in \Sigma_1 \). Thus we get the following

**Corollary.** If \( M \) and \( \tilde{M} \) are canonical and \( \text{Ind}(M) \in \Sigma_2 \cap \Pi_2 \), then there is a non-empty set \( L \) completely included in \( M \) or in \( \tilde{M} \) with \( \text{Ind}(L) \in \Sigma_1 \).

**\( \Sigma_2 \)-completeness of index sets**

A set \( A \) is \( \Sigma^B_n \)-complete if \( A \in \Sigma^B_n \) and \( \forall C (C \in \Sigma^B_n \Rightarrow C \equiv_1 A) \). \( \Pi^B_n \)-completeness is defined similarly. According to exercise 14-10 in the book of Rogers [9] every \( \Sigma^B_n \)-complete set forms a 1-degree which is also an \( m \)-degree. Hence we get an equivalent definition replacing one–one reducibility (\( \equiv_1 \)) by many–one reducibility (\( \equiv_m \)).
Theorem 6. If $M$ is canonical then the following statements are equivalent:

(i) $\text{Ind}(M)$ is $\Sigma_2^B$-complete.
(ii) $\text{Ind}(M) \in \Sigma_2^B \setminus \Pi_2^B$.
(iii) $\text{Ind}(M) \in \Sigma_2^B$ and $\text{Approx}(M) \notin M$.

Proof. Clearly, (i) implies (ii).

If $\text{Ind}(M) \in \Sigma_2^B \setminus \Pi_2^B$, then $M \subseteq \text{Approx}(M)$ by Lemma 2(ii) and therefore $\text{Approx}(M) \notin M$ (otherwise $\text{Ind}(M) \in \Pi_2^B$ by Lemma 3). Thus (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) follows directly by Lemma 1(i). $\square$

The topic of the last part of this paper is to present index sets in $\Sigma_2^B \setminus \Pi_2^B$ which are not $\Sigma_2^B$-complete. To do this we need some preparations. For every set $M$ of r.e. sets $\text{Ind}(M)_{\text{min}} = \{ i \mid W_i \in M \land \forall j < i (W_i \neq W_j) \}$ is the set of all minimal indices for sets in $M$.

The following result is due to M. Blum, probably unpublished. Reference is made to it in [7].

Lemma 7. If $M$ is infinite, then $\text{Ind}(M)_{\text{min}}$ is immune.

Proof. According to Theorem 1 in [1] for every recursive function $g$ with infinite range there exist $i, j$ such that $\phi_i = \phi_{g(j)}$ and $i < g(j)$. Supposing that $\text{Ind}(M)_{\text{min}}$ is not immune we can choose $g$ such that $g(\mathbb{N}) \subseteq \text{Ind}(M)_{\text{min}}$. But then $\phi_i = \phi_{g(j)}$ (i.e., $W_i = W_{g(j)}$) and $i < g(j)$ for some $i$ and $j$ means that $g(j) \notin \text{Ind}(M)_{\text{min}}$ — a contradiction. $\square$

Lemma 8. If $\text{Ind}(M) \in \Sigma_2^B \setminus \Pi_2^B$, then

(i) $\text{Ind}(M)_{\text{min}} \in \Sigma_2^B \setminus \Pi_2^B$, and
(ii) $\text{Ind}(M)_{\text{min}}$ is immune.

Proof. (i) The fact that $\text{Ind}(M)_{\text{min}} \in \Sigma_2^B$ follows directly from the assumption $\text{Ind}(M) \in \Sigma_2^B$ by the Tarski–Kuratowski algorithm [9] since

$$i \in \text{Ind}(M)_{\text{min}} \iff i \in \text{Ind}(M) \land \forall j < i (W_i \neq W_j)$$

where $i \in \text{Ind}(M) \in \Sigma_2^B$ and

$$\forall j < i (W_i \neq W_j) \iff \forall j \exists x ((x \in W_i \land x \notin W_j) \lor (x \in W_j \land x \notin W_i))$$

which is in $\Sigma_2$ and hence in $\Sigma_2^B$ by standard Tarski–Kuratowski manipulations.

Similarly

$$i \in \text{Ind}(M) \iff \exists j \leq i (j \in \text{Ind}(M)_{\text{min}} \land W_i = W_j)$$
where
\[ W_i = W_j \iff \forall x \ ((x \in W_i \land x \in W_j) \lor (x \notin W_i \land x \notin W_j)) \]
is in \( \Pi_2 \) and hence in \( \Pi_2^B \).

If \( \text{Ind}(M)_{\text{min}} \in \Pi_2^B \) as well, then \( \text{Ind}(M) \in \Pi_2^B \) by the Tarski–Kuratowski algorithm, contradicting the hypothesis that \( \text{Ind}(M) \in \Sigma_2^B \setminus \Pi_2^B \).

(ii) Observe that \( M \) must be infinite; otherwise \( \text{Ind}(M)_{\text{min}} \) is finite contradicting the fact that \( \text{Ind}(M)_{\text{min}} \in \Sigma_2^B \setminus \Pi_2^B \). Thus \( \text{Ind}(M)_{\text{min}} \) is immune by Lemma 7. □

In the following let \( B' = \{ x \mid x \in W_x^B \} \) and \( B'' = (B')' \). Furthermore we denote by \( \text{Ind}_B(M) = \{ i \mid W_i^B \in M \} \) the \( B \)-index set of \( M \). In the proof of the following two lemmata we use the fact that \( \text{Ind}_B(E) =_{m} B'' \) and \( \text{Ind}_B((\emptyset)) =_{m} B' \). The two statements can be proved by replacing in the proofs for \( \text{Ind}(E) =_{m} \emptyset'' \) and \( \text{Ind}(\emptyset) =_{m} \emptyset' \) the partial recursive functions by the partial \( B \)-recursive functions. We skip the proof here. Finally we define for every set \( B \), \( S_B = \{ W \mid W \text{ r.e. and } W \subseteq B \} \) as the set of all r.e. subsets of \( B \) and \( FS_B = \{ W \mid W \text{ finite and } W \subseteq B \} \) as the set of all finite subsets of \( B \). It is easily verified that \( B \leq_m \text{Ind}(S_B) \) and \( B \leq_m \text{Ind}(FS_B) \).

**Lemma 9.** \( \text{Ind}(S_B') =_{m} B'' \).

**Proof.** This is an immediate consequence of the well-known fact that if \( S \) is \( \Sigma_2 \)-complete, then \( \{ i \mid W_i \subseteq S \} \) is \( \Pi_2^{\text{non}} \)-complete (see e.g. the note after Theorem 6.3 of [5]). □

**Lemma 10.** If \( C \leq_m B'' \), then \( \text{Ind}(FS_C) \leq_m B'' \).

**Proof.** By the above observation \( C \leq_m B'' =_{m} \text{Ind}_B(E) \). Hence there exists a recursive function \( f \) with \( f^{-1}(\text{Ind}_B(E)) = C \). Define by the relativized \( s'' \)-theorem a recursive function \( \alpha(i) \) as the index of the \( B \)-recursively enumerable set defined by
\[ W_{\alpha(i)} = W_i \cup \bigcup_{x \in f(W_i)} W_x^B. \]
We show \( \alpha^{-1}(\text{Ind}_B(E)) = \text{Ind}(FS_C) \), i.e., \( \text{Ind}(FS_C) \leq_m \text{Ind}_B(E) =_{m} B'' \) and the statement follows immediately.

If \( i \in \text{Ind}(FS_C) \), then \( W_i \) is finite and \( W_i \subseteq C \). Hence \( f(W_i) \) is finite and \( f(W_i) \subseteq \text{Ind}_B(E) \), i.e., \( W_x^B \) is finite for every \( x \in f(W_i) \). Since a finite union of finite sets is a finite set, \( W_{\alpha(i)}^B \) is finite, i.e., \( \alpha(i) \in \text{Ind}_B(E) \).

If \( i \notin \text{Ind}(FS_C) \), then (1) \( W_i \) is infinite or (2) \( W_i \notin C \). Clearly, if \( W_i \) is infinite, then \( W_{\alpha(i)}^B \) is infinite, too, and we get \( \alpha(i) \notin \text{Ind}_B(E) \). In the second case we get \( f(W_i) \notin \text{Ind}_B(E) \). Hence there exists some \( x \in f(W_i) \) such that \( W_x^B \) is infinite, i.e., \( W_{\alpha(i)}^B \) is infinite, and we get \( \alpha(i) \notin \text{Ind}_B(E) \) in this case, too. □
Now we shall show that $B' = _m \text{Ind}(S_C)$ if $C$ is immune. Observe that $S_C = FS_C$ for every immune set $C$. This fact will be the key to the proof.

**Lemma 11.** If $C$ is immune, then $B' = _m \text{Ind}(S_C)$.

**Proof.** We proceed by contradiction. Suppose that there exist an immune set $C$ and some $B \subseteq \mathbb{N}$ such that $B' = _m \text{Ind}(S_C)$. We shall show that the assumption yields $\text{Ind}(S_B) \leq _m \text{Ind}(FS_B)$.

Since $C \leq _m \text{Ind}(S_C) = _m B'$, there exist recursive functions $f$ and $g$ with $f^{-1}(B') = C$ and $g^{-1}(\text{Ind}(S_C)) = B'$. With the help of the $s^m_n$-theorem define a recursive function $\beta(i)$ to be the index of the r.e. set defined by

$$W_{\beta(i)} = \bigcup_{x \in g(W_i)} f(W_x).$$

We show $\beta^{-1}(\text{Ind}(FS_B)) = \text{Ind}(S_B)$.

If $i \in \text{Ind}(S_B)$, then $W_i \subseteq B'$. Hence $g(W_i) \subseteq \text{Ind}(S_C)$, i.e., $\bigcup_{x \in g(W_i)} W_x \subseteq C$. Since the union of the $W_x$ is r.e. and $C$ is immune, the union of the $W_x$ must be a finite set. Thus

$$W_{\beta(i)} = \bigcup_{x \in g(W_i)} f(W_x) = f\left(\bigcup_{x \in g(W_i)} W_x\right)$$

is finite.

$\bigcup_{x \in g(W_i)} W_x \subseteq C$ implies $W_{\beta(i)} \subseteq f(C)$. Since $f(C) \subseteq B'$ we get $W_{\beta(i)} \subseteq B'$ and $W_{\beta(i)}$ is finite. Thus $\beta(i) \in \text{Ind}(FS_B)$.

If $i \notin \text{Ind}(S_B)$, then $W_i \notin B'$. Hence $g(W_i) \notin \text{Ind}(S_C)$, i.e., there exists $x \in g(W_i)$ such that $W_x \notin C$ and therefore $f(W_x) \notin B'$. Thus $W_{\beta(i)} \notin B'$, i.e., $\beta(i) \notin \text{Ind}(FS_B)$.

In summary $\text{Ind}(S_B) \leq _m \text{Ind}(FS_B)$. Applying Lemmas 9 and 10 yields $B'' = _m \text{Ind}(S_B) \leq _m \text{Ind}(FS_B) = _m B''$—a contradiction to the fact that $B''$ and $B''$ are incomparable with respect to many-one reducibility. ∎

Combining Lemmas 8 and 11 we get

**Theorem 12.** If $\text{Ind}(M) \in \Sigma^B_2 \setminus \Pi^B_2$, then

(i) $\text{Ind}(S_C) \in \Sigma^B_2 \setminus \Pi^B_2$, and

(ii) $\text{Ind}(S_C)$ is not $\Sigma^B_2$-complete, where $C = \text{ind}(M)_{\text{min}}$.

**Proof.** (i) $C$ is immune and $C \in \Sigma^B_2 \setminus \Pi^B_2$ by Lemma 8. Hence $\text{Ind}(S_C) = \text{Ind}(FS_C) \leq _m B'' = \Sigma^B_2$ by Lemma 10. Furthermore $\text{Ind}(S_C) \notin \Pi^B_2$ since $C \leq _m \text{Ind}(S_C)$ and $C \notin \Pi^B_2$.

(ii) Follows immediately by Lemma 11 (otherwise $(B')' = _m \text{Ind}(S_C)$). ∎

By Theorem 12 we get index sets on every level >1 of the arithmetical
hierarchy which are not complete on that level. To see this fix some \( n \geq 2 \) and define \( M_n = FS_{\emptyset^{(n)}} \), where \( \emptyset^{(n)} \) is the \( n \)-th jump of \( \emptyset \) defined as in [9]. Then \( \text{Ind}(M_n) = \emptyset^{(n)} \) by Lemma 10, i.e., \( \text{Ind}(M_n) \in \Sigma_n \setminus \Pi_n \). Define \( B = \emptyset^{(n-2)} \). Then \( \text{Ind}(M_n) \in \Sigma_2^B \setminus \Pi_2^B \) so that by Theorem 12, \( \{ i \mid W_i \subseteq \text{Ind}(M_n)_{\text{min}} \} \) is an index set in \( \Sigma_2^B \setminus \Pi_2^B = \Sigma_n \setminus \Pi_n \) which is not \( \Sigma_n \)-complete. Accordingly the complement \( \{ i \mid W_i \notin \text{Ind}(M_n)_{\text{min}} \} \) is an index set in \( \Pi_n \setminus \Sigma_n \) which is not \( \Pi_n \)-complete.

Acknowledgements

I wish to thank the unknown referee for his suggestions and improvements.

References