The Cocke-Younger-Kasami Algorithm
- Revised -

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Abstract

The wellknown algorithm of Cocke-Younger-Kasami, solving the wordproblem for contextfree grammars in Chomsky-Normalform in time $O(|w|^3)$ with the help of the recognition matrix can be extended to arbitrary contextfree grammars. The resulting time bound is $O(|w|^{2+(||G||-1)})$ where $||G||$ is a very natural number associated to G. Moreover for linear grammars we get time $O(|w|^2)$, the bound from Earley's algorithm, and with small variations $O(|w|)$ for one-sided-linear grammars.

Keywords: wordproblem, contextfree grammars, recognition matrix, time-complexity

Introduction

The starting point for this note is the simple observation

$$w \in L(G) \Longleftrightarrow \{w\} \cap L(G) \neq \emptyset$$

(G a (contextfree) grammar and $L(G)$ the generated language).

This is the reduction from the wordproblem to the emptiness-problem.

Since $\{w\}$ is a regular set with a very special minimal Rabin-Scott-acceptor, one can use the wellknown triple construction for the intersection-theorem.

Rewriting this triple construction into the recognition-matrix, we can avoid under special circumstances both the construction via the intersection-theorem and the design of a good algorithm for the emptiness problem.

Furthermore we can avoid the transformation of the original grammar into some normalform, especially we need not prepare for erasing and chaining. A reasonable timebound results, giving the timebound of Cocke-Younger-Kasami in the case of Chomsky-Normalform.

Moreover, for linear grammars the quadratic timebound of Earley's algorithm results. With a small variation of the basic algorithm we get for one-sided linear grammars (regular grammars) a linear timebound, as it should be.

Besides the knowledge, that by different approach we get the timebound $O(|w|^3)$, our result may be of didactic value due to the simplicity of the argument.

1 Notations

If X is an **alphabet**, then X^* is the free monoid with the **empty** word $\Box$.

Consider $X' \subseteq X$. Let $w \in X^*$, then w has a unique decomposition

$$\begin{aligned} w &= w_o x'_1 w_1 \cdots x'_r w_r \text{ with} \\ & \quad x'_i \in X' (1 \leq i \leq r) \text{ and } w_i \in (X \setminus X')^* (0 \leq i \leq r). \end{aligned}$$

We call it the **X′ -decompostion.**

Denote by $|w|_{X'} = r$ the length with respect to X'. Obviously, $|w|_X = |w|$ is the **length** of w.

If $w = x_1 \dots x_n$ with $x_m \in X$ for $1 \leq m \leq n$ and $0 \leq i \leq j \leq n$ denote by $w[i,j]$ the word

$$w[i,j] = x_{i+1} \dots x_n \text{ if } 0 < i < j \text{ and } w[i,i] = \Box.$$

For $j < i$ $\quad w[i,j]$ is undefined.

Note: $w[i-1,i] = x_i \qquad 1 \leq i \leq n.$

A **grammar** G is a quadruple $G = (\sigma, Z, T, P)$, where
- Z is the alphabet of **variables**
- T is the alphabet of **terminals**
- $\sigma \in Z$ is the **startsymbol**
- $P \subseteq Z \times (Z \cup T)^*$ is the (finite) set of **productions.**

A rule $r \in P$ is usually written in the form $r = (p \to q)$.

By $u \vdash v$ we denote the **direct-derivation** from u to v, $u \vdash v$ is the transitiv and reflexive closure of $\vdash$.

The **generated language** of G is therefore defined by

$$L(G) = \{w \in T^* | \quad \sigma \overset{*}{\vdash} w\}.$$

In this paper we are interested only in **contextfree** grammars and **contextfree** languages, just defined by contextfree grammars, i.e. for all $p \to q \in P \qquad p \in Z$ holds.

More details on grammars and languages can be found in standard textbooks like [1] & [2] for example.

We introduce a measure for grammars G by

$$||G|| = Max\{|q|_z| \quad \exists \quad p \to q \in P\}.$$

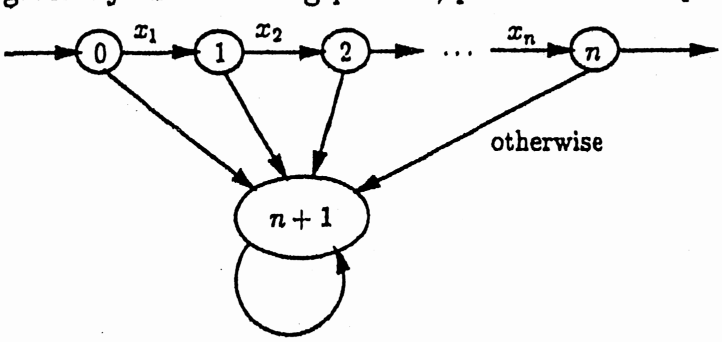
For example, a **linear** grammar G is a contextfree grammar with $||G|| \leq 1$.

2 Preparations

The basic idea is the following connection between languages $L \subseteq X^*$ and $w \in X^*$:

$$w \in L \Longleftrightarrow L \cap \{w\} \neq \emptyset.$$

$\{w\}$ is a **regular** set (see [1], [2]). The minimal Rabin-Scott-acceptor is given by the following picture, provided $w = x_1 \dots x_n (x_i \in X, 1 \leq i \leq n)$:



where 0 is the **initial** state, n is the **accepting** state and $n+1$ is the **fault** state.

If δ is the **transition-function** and δ^* the extension of δ to words we immediately see:

$$\delta^*(u, i) = j \Longleftrightarrow u = w[i, j] \qquad (0 \leq i \leq j \leq n).$$

Consider a contextfree grammar G and a word $w \in T^*$. Now we can use a modified triple-construction to create a grammar G_w with

$$L(G_w) = L(G) \cap \{w\},$$

hence we have reduced the **wordproblem** to the **emptiness-problem** for contextfree grammars (see [1] for details). The modifications are elaborated in the way that terminal parts of the right-hand-side of a rule are processed directly.

The Rabin-Scott-acceptor for $\{w\}$ has special properties (some kind of monotonicity for example). Therefore it is not necessary to construct G_w explicitly.

We make use of the **recognition-matrix** $T_{w,G}$ (see [1]).

This is a matrix of format$(n+1, n+1)$, where numeration of columns and rows start with 0 instead of the usual 1. It is defined for arbitrary grammars by

$$T_{w,G}[i,j] = \{\xi \in Z \quad | \quad \xi \stackrel{*}{\vdash} w[i,j]\} \qquad (0 \leq i,j \leq n).$$

$T_{w,G}$ is an upper triangular matrix.

The criterium for "$w \in L(G)$?" can be rewritten in the form $\sigma \in T[0,n]$.

Since we use a modified version of the triple-construction we need a prepreprepared table of statetransitions for the terminal parts of the grammar G.

Let $r = (p \to q) \in P$ and $q = u_o\xi_1 \ldots \xi_s u_s$ be the Z-decomposion of q, then

$$Terminal(r) = \{u_i \quad | \quad 0 \leq i \leq s\}$$
$$\text{and}$$
$$Terminal(G) = \bigcup_{r \in P} Terminal(r)$$

We prepare a table Δ for all transitions

$$\delta^*(u,i) = j \quad (u \in Terminal(G), 0 \leq i,j \leq n+1).$$

This table is of format $(n+2, (||G||+1) \cdot \sharp(P))$.

On a **Random-Access-Machine** (see[1]), the length of $w(=n)$ must be part of the input, hence addressing an entry of Δ takes constant time. Given $w, |w|$ and G the preparation of Δ needs linear time on a RAM.

Example:

$$G : \sigma \to (\sigma)\sigma \,|\, () \,|\, \Box$$

generating the Dyck-language D_1 (see [1]). Let $w = (()())()$, we get $|w| = 8$ and Δ is given by

	0	1	2	3	4	5	6	7	8	9
$\Box$	0	1	2	3	4	5	6	7	8	9
(	1	2	9	4	9	9	9	9	9	9
)	9	9	3	9	5	6	9	8	9	9
()	9	9	4	9	6	9	8	9	9	9

Special treatment has to be given to the processes of **erasing** and **chaining** in contextfree grammars.

Define for any $Z' \subseteq Z$

$$\mathrm{Chain}(Z') = \{\xi \mid \exists \eta \in Z' : \xi \overset{*}{\vdash} \eta\}.$$

This operation can be done in constant time and can be preprepared.

Observation 1:

Chain is a closure-operator, i.e.

(1) $Z' \subseteq \mathrm{Chain}(Z')$ for $Z' \subseteq Z$
(2) $Z' \subseteq Z'' \subseteq Z \Longrightarrow \mathrm{Chain}(Z') \subseteq \mathrm{Chain}(Z'')$
(3) $Z' \subseteq Z \Longrightarrow \mathrm{Chain}(\mathrm{Chain}(Z')) = \mathrm{Chain}(Z')$
(4) $\mathrm{Chain}(\emptyset) = \emptyset$
(5) $\mathrm{Chain}(Z) = Z$.

Observation 2:

Let $T(w) = \{\xi \mid \xi \overset{*}{\vdash} w\}$ then

$\mathrm{Chain}(T(w)) = T(w)$ and therefore

$\mathrm{Chain}(T_{w,G}[i,j]) = T_{w,G}[i,j]$ for all $0 \leq i,j \leq n$.

3 The algorithm

To compute $T_{w,G}$ we start with the initialization.

Observation 3:

(1) For all $1 \leq i \leq n$: $\quad T_{w,G}[i,i] = T(\Box)$

(2) For all $0 \leq i \leq j \leq n$:

$$\text{Chain}(\{\xi \mid \exists u \in T^* : u = w[i,j] \text{ and } \xi \rightarrow u \in P\}) \subseteq T_{w,G}[i,j]$$

Therefore we can initialize in the following way:

for $i = 0$ **to** n **do** $T_{w,G}[i,i] := T(\Box)$ **od**

for $i = 0$ **to** n **do**

for $j = i+1$ **to** n **do**

$T_{w,G}[i,j] :=$ Chain$(\{\xi | \exists u \in T^* : u = w[i,j]$ **and** $\xi \rightarrow u \in P\})$

od

od

The time costs are $O(|w|)$ for the first loop and $O(|w|^2)$ for the second and the third loop, in summary $O(|w|^2)$, since the internal operations take constant time. The complexity is measured on a RAM.

Example 1:

Consider the grammar G given by

$$\sigma \rightarrow (\sigma)\sigma \quad | \quad (\ ) \quad | \quad \Box \text{ and } w = (()())()$$

After initialization the current value of $T_{w,G}$ is

	0	1	2	3	4	5	6	7	8	
0	σ	$\emptyset$	$\emptyset$							
1		σ	$\emptyset$	σ						
2			σ	$\emptyset$	$\emptyset$					
3				σ	$\emptyset$	σ				
4					σ	$\emptyset$	$\emptyset$			
5						σ	$\emptyset$	$\emptyset$		
6							σ	$\emptyset$	σ	
7								σ	$\emptyset$	
8									σ	

All other entries are $= \emptyset$.

Example 2:

Consider the grammar

$\sigma \to \xi\sigma\xi \;\; | \, c \, | \;\; \square$

$\xi \to a\xi b \;\; | \;\; \square$ and $w = a^2b^2cab$

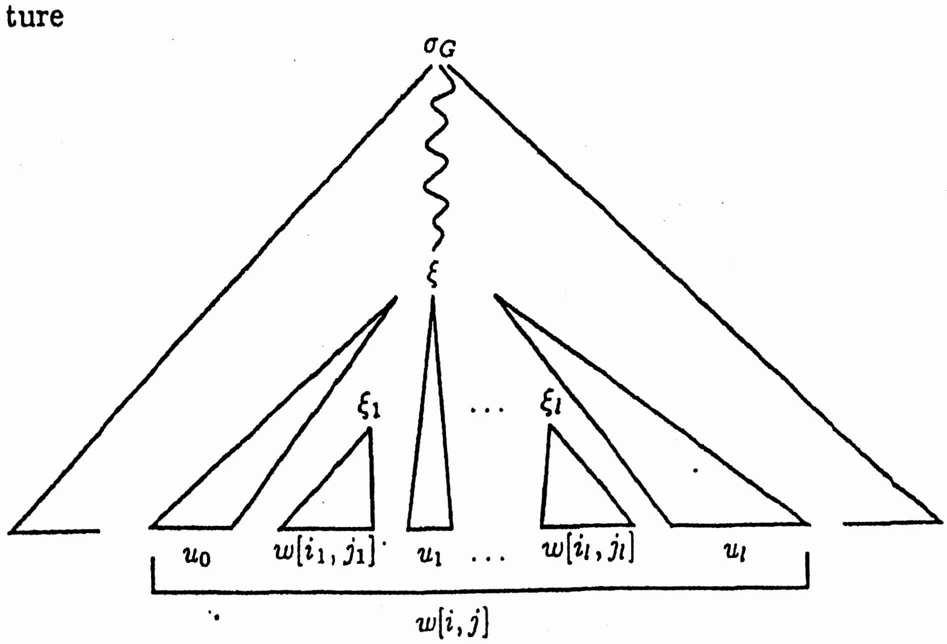
$T(\square) = \{\xi\}$, $\mathrm{Chain}(\xi) = \{\sigma, \xi\}$, $\mathrm{Chain}(\sigma) = \{\sigma\}$

After initialization the current value of $T_{w,G}$ is

	0	1	2	3	4	5	6	7
0	σ,ξ	$\emptyset$						
1		σ,ξ	$\emptyset$					
2			σ,ξ	$\emptyset$				
3				σ,ξ	$\emptyset$			
4					σ,ξ	σ		
5						σ,ξ	$\emptyset$	
6							σ,ξ	$\emptyset$
7								σ,ξ

All other entries are $= \emptyset$.

The whole computation of T_w can be easily derived from the following picture



with the following conditions:

- $\xi \to u_0\xi_1 u_1 \dots \xi_l u_l \in P$
- $\delta^*(u_o, i) = i_1, \delta^*(u_l, j_l) = j$
- $\delta^*(u_\lambda, j_\lambda) = i_{\lambda+1} \qquad 0 < \lambda < l$
- $\xi_\lambda \in T_{w,G}[i_\lambda, j_\lambda]$
- $i \leq j_1 \leq j_2 \leq \dots \leq j_{l-1} \leq j_l \leq j$
- $l \leq ||G||$.

Therefore we get the following algorithm after initialization

for $i = 0$ **to** n **do**

for $j = i + 2$ **to** n **do**

$T_{w,G}[i,j] :=$ Chain$(T_{w,G}[i,j]$
$\cup\{\xi \in Z \;\;|\;\; \exists l \geq 1, i \leq j_1 \leq j_2 \leq \dots \leq j_l \leq j$ and $\xi \to q \in P$
with Z-decomposition $q = u_0\xi_1 \dots \xi_l u_l$:

(1) $\exists 1 \leq \lambda \leq l : i < j_\lambda < j$
(2) $l = 1 \Longrightarrow u_0 u_1 \neq \Box$
(3) $\xi_1 \in T_{w,G}[\delta^*(u_0, i), j_1]$
(4) $\delta^*(u_l, j_l) = j$
(5) $\xi_\lambda \in T_{w,G}[\delta^*(u_{\lambda-1}, j_{\lambda-1}), j_\lambda] \quad (1 < \lambda \leq l)\})$
od
od

The criterion of success is simply

$$\sigma \in T_{w,G}[0, n].$$

Example 1:

Consider the grammar $G : \sigma \to (\sigma)\sigma|()|\Box$ and the word $w = (()())()$.
We compute $T_{w,G}[1,5], w[1,5] = ()()$. The only production which can be used is $\sigma \to (\sigma)\sigma$. The only choice for the j_λ is the following:

$$j_1 = 2, \text{ since } \delta^*((,1) = 2 \text{ and } \sigma \in T_{w,G}(2,2) = \{\sigma\}$$

$$j_2 = 5, \text{ since } \delta^*(),2) = 3 \text{ and } \sigma \in T_{w,G}(3,5) = \{\sigma\} \text{ and}$$

$$\delta^*(\Box, 5) = 5$$

hence $\sigma \in T_{w,G}[1,5]$.

The chain operation is useless in this case.

The resulting recognition-matrix is

	0	1	2	3	4	5	6	7	8
0	σ	$\emptyset$					σ		σ
1		σ	$\emptyset$	σ		σ			
2			σ	$\emptyset$	$\emptyset$				
3				σ	$\emptyset$	σ			
4					σ	$\emptyset$	$\emptyset$		
5						σ	$\emptyset$	$\emptyset$	
6							σ	$\emptyset$	σ
7								σ	$\emptyset$
8									σ

The other entries are $= \emptyset$.

Therefore (( ) ( )) ( ) $\in L(G)$

Example 2:

Consider $G : \sigma \to \xi\sigma\xi \mid c \mid \square$ and $\xi \to a\xi b \mid \square$ and a^2b^2cab

- $T_{w,G}[0,2]$. We have two possible rules

(1) $\sigma \to \xi\sigma\xi$ i.e. $l = 3, u_0 = u_1 = u_2 = u_3 = \square \Longrightarrow j_3 = 2$.
By definition(1) either $j_1 = 1$ or $j_2 = 1$.
If $j_1 = 1$ then $\xi \in T_{w,G}[0,1] = \emptyset$, a contradiction. If $j_2 = 1$ then $\xi \in T_{w,G}[1,2] = \emptyset$, again a contradiction.

(2) $\xi \to a\xi b, i.e.\ l = 1, u_o = a, u_1 = b \Longrightarrow j_1 = 2$, but $\delta^*(2,b) = 3$
a contradiction
In summary $T_{w,G}[0,2] = \emptyset$.

- $T_{w,G}[1,3]$. Again two possible rules

(1) $\sigma \to \xi\sigma\xi$, i.e. $l = 3, u_0 = u_1 = u_2 = u_3 = \square$
$\Longrightarrow j_3 = 3$, again either $j_1 = 2$ or $j_2 = 2$.
In both cases we get a contradiction.

(2) $\xi \to a\xi b$, i.e. $l = 1, u_0 = a, u_1 = b, j_1 = 2$ and $\xi \in T_{w,G}[2,2]$
hence $\xi \in T_{w,G}[1,3]$.

By Chain we get $T_{w,G}[1,3] = \{\sigma, \xi\}$

- $T_{w,}[0,4]$. Again two possible rules

(1) $\sigma \to \xi\sigma\xi, i.e.\ l = 3, u_0 = u_1 = u_2 = u_3 = \Box \Longrightarrow j_3 = 4$

 (i) $j_1 = 1$ impossible

 (ii) $j_1 = 2$ impossible

 (iii) $j_1 = 3 \Longrightarrow j_2 = 3$ or $j_2 = 4$
 In the first case $\xi \in T_{w,G}[3,4]$,
 in the second case $\sigma \in T_{w,G}[3,4]$.

 (iv) $j_2 = 2$ or $j_2 = 1$ or $j_2 = 3$ analogously.

(2) $\xi \to a\xi b, i.e.\ l = 1, u_0 = a, u_1 = b, j_1 = 3,$
$\xi \in T_{w,G}[1,3] = \{\sigma, \xi\} \Longrightarrow \xi \in T_{w,G}[0,4]$

By Chain we get $T_{w,G}[0,4] = \{\sigma, \xi\}$.

Remarks:

- The exclusion of chain-rules by condition (2) is compensated by the Chain-operation.
- By condition (1) we get $j_\lambda - j_{\lambda-1} < j - i \quad (2 \leq i \leq l)$,
 together with condition $T_{w,G}[i,i] = T(\Box)$ for all $0 \leq i \leq n$, we can organize the algorithm in an ON-LINE-mode. Our version is OFF-LINE.
- Knowing the recognition-matrix it should be easy to construct a parser without increasing time-complexity.

We now turn our interest to time-complexity. Observe, that j_l – if existent – is uniquely determined by j and u_l (Condition(4)).

Hence, we have "free" choices for $j_1, j_2, \ldots, j_{l-1}$. These leads to $l-1$ loops. The crucial condition (1) can be checked by a boolean variable in the body of the loops.

Worst-case-bounds are $0 \leq j_\lambda \leq n \quad (1 \leq \lambda \leq l-1)$,

$l \leq ||G||$ and $0 \leq i, j \leq n$.

Hence, we get

$$O(n^2 \cdot n^{l-1}) = O(n^{2+(||G||-1)})$$

as the overall worst-case-time-bound, provided $||G|| \neq 0$.

Since prepreparation and initialization have time-bounds $O(n)$ and $O(n^2)$ resp., we get in whole the time-bound

$$O(|w|^{2+(||G||-1)}).$$

4 Special cases

I. Normalforms:

For a grammar G in **Chomsky-normalform** all productions are of the form

$\xi_0 \to \xi_1\xi_2 \quad (\xi_{0,1,2} \in Z)$ or
$\xi_0 \to t \quad (\xi_0 \in Z, t \in T)$, hence

$||G|| = 2$ and therefore time-complexity is $0(|w|^3)$. Indeed, in this case the Cocke-Younger-Kasami-algorithm results.

For a grammar G in **2-Greibach-normalform** all productions are of the form

$\xi_0 \to t\xi_1\xi_2 \quad (\xi_{0,1,2} \in Z, t \in T)$ or

$\xi_0 \to t\xi_1 \quad (\xi_{0,1} \in Z, t \in T)$ or

$\xi_0 \to t \quad (\xi_0 \in Z, t \in T)$, hence

$||G|| = 2$ and therefore time-complexity is $O(|w|^3)$, giving the same result as in the Chomsky-normalform-case.

II. Linear grammars

Recall, a contextfree grammar is **linear** iff $||G|| \leq 1$, hence we get the time-complexity $O(|w|^2)$, which is the bound of Earley's algorithm, and is not reached by the Cocke-Younger-Kasami-algorithm, without altering the algorithm.

III. One-sided linear grammars

In a **rightlinear** grammar all productions are of the form $\xi_0 \to u\xi_1$ with

$$\xi_0 \in Z, \xi_1 \in Z \cup \Box \text{ and } u \in T^*.$$

In this case $j_1 = n$, the "target" state. Therefore we only have to compute $T_{w,G}[n,n], \ldots, T_{w,G}[0,n]$, knowing that $T_{w,G}[n,n] = T(\Box)$.

Therefore, both phases -initialization and computation - can be simplified drastically.

The resulting algorithm is:

Initialization:

for $i = n$ **downto** 0 **do**

$T_{w,G}[i,n] :=$ Chain$(\{\xi \in Z \quad | \quad \exists \xi \to u \in P \text{ with } u = w[i,n]\})$ **od**

Computation:

for $i = 0$ **to** n **do**

$T_{w,G}[i,n] := \text{Chain}(\{\xi \in Z \mid \exists u \in T^*, \eta \in Z :$
$\eta \in T_{w,G}[\delta^*(u,i), n]$ and $\xi \to u\eta \in P\})$

od

Obviously, the time-complexity is $O(|w|)$ - as it should be.

The same kind of simplification can be used for leftlinear grammars, where all productions are of the form

$$\xi_0 \to \xi_1 u \text{ with } \xi_0 \in Z, \xi_1 \in Z \cup \square \text{ and } u \in T^*.$$

In this case the " source " state 0 is fixed, hence we only have to compute $T_{w,G}[0,0], \ldots, T_{w,G}[0,n]$.

Therefore we get $O(|w|)$ as time-complexity-bound again.

Note, we do not need any normalform or a reduction to deterministic Rabin-Scott-acceptors to get the result.

5 Concluding remarks

We haven't discussed, wether it is possible to use some kind of Valiant-type reductions via interpreting $T_{w,G}$ as a " closure " and then reducing the computing of this closure to Boolean matrix-multiplication.

6 References

All what we used in this note is very familiar to those knowing the basics of formal language theory. Therefore two references will suffice

[1] M. Harrison, Introduction to Formal Language Theory, Addison-Wesley Pub.Co., Reading Mass., 1978

[2] G. Rozenberg - A. Salomaa, Handbook of Formal Languages, three volumes, Springer Verlag Berlin, Heidelberg, New York, 1997