MIP Solver

Main Components:

- Preprocessing / Structure Analysis
- (Efficient) Branch-and-Bound
- Cutting Plane Generation
- Primal Heuristics
- (Fast) LP solvers

Available Software:

- `ABACUS` (University of Cologne, Germany)
- `bc-opt` (CORE, Belgium)
- `CPLEX` (ILOG, France)
- `LINDO` (LINDO Systems, USA)
- `MIPO` (Columbia University, USA)
- `MINTO` (Georgia Institute of Technology, USA)
- `MPS III` (Ketron Management Science, USA)
- `MPSX` (IBM, USA)
- `OSL` (IBM, USA)
- `SIP` (TU Darmstadt and ZIB, Germany)
- `XA` (Sunset Software Technology, USA)
- `XPRESS-MP` (Dash, UK)
- .....

➥ Saltzman: *"Survey: Mixed Integer Programming"*, OR/MS Today, April 1994

Preprocessing:

- Eliminate redundant information
- Strengthen formulation by logical implications

For some special problems very effective:

- Steiner tree problems
  Duin, Volgenant'89, Duin'97, Koch, Martin'98, ...
- Set packing / partitioning problems
  Borndörfer'98, Garfinkel, Nemhauser'69, Hoffmann, Padberg'93, ...

For MIPs ...

- Andersen, Andersen'95
- Bixby'94
- Crowder, Johnson, Padberg'83
- Hoffman, Padberg'91
- Suhl, Szymanski'94
- .....

Consider

$$\begin{aligned} \text{min} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & l \leq x \leq u \\ & x \in \mathbb{Z}^{n-p} \times \mathbb{R}^p, \end{aligned}$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $c, l, u \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

Preprocessing:

- Empty / Parallel Rows
- Empty / Infeasible / Fixed Columns
- Singleton Columns / Rows
- Forcing and Dominated Rows

Determine for row i

$$L_i = \sum_{j:a_{ij}>0} a_{ij} l_j + \sum_{j:a_{ij}<0} a_{ij} u_j$$

$$U_i = \sum_{j:a_{ij}>0} a_{ij} u_j + \sum_{j:a_{ij}<0} a_{ij} l_j$$

Then

$$\begin{array}{lcl} L_i > b_i & \Rightarrow & \text{infeasible row} \\ L_i = b_i & \Rightarrow & \text{forcing row} \\ U_i < b_i & \Rightarrow & \text{dominated row} \end{array}$$

- Strengthen variable bounds

$$x_j \le \begin{cases} (b_i - L_i)/a_{ij} + l_j, & \text{if } a_{ij} > 0 \\ (L_i - U_i)/a_{ij} + l_j, & \text{if } a_{ij} < 0 \end{cases}$$

$$x_j \ge \begin{cases} (L_i - U_i)/a_{ij} + u_j, & \text{if } a_{ij} > 0 \\ (b_i - L_i)/a_{ij} + u_j, & \text{if } a_{ij} < 0. \end{cases}$$

Example (from `Miplib`-problem mod015)

$-45x_6 - 45x_{30} - 79x_{54} - 53x_{78} - 53x_{102} - 670x_{126} \le -443$,

where $x_j \in \{0,1\}$. Then, $L_i = -945, U_i = 0$, and

$$x_j \ge (-443 + 945)/-670 + 1 = 0.26.$$

Preprocessing:

- Coefficient Reduction

Example (from Miplib-problem p0033)

(1) $$-230x_{10} - 200x_{16} - 400x_{17} \leq -5$$

Replace (1) by

$$-5x_{10} - 5x_{16} - 5x_{17} \leq -5$$

In general, if for $x_j \in \{0, 1\}$

$$\begin{array}{lll} a_{ij} < 0 \text{ and } U_i + a_{ij} < b_i, & \text{set} & a'_{ij} = b_i - U_i, \\ a_{ij} > 0 \text{ and } U_i - a_{ij} < b_i, & \text{set} & \left\{ \begin{array}{l} a'_{ij} = U_i - b_i, \\ b_i = U_i - a_{ij}, \end{array} \right. \end{array}$$

where a'_{ij} denotes the new reduced coefficient.

- Aggregation

Use equations to eliminate variables.
For instance, use

$$a_{ij}x_j + a_{ik}x_k = b_i$$

to replace x_k by

$$\frac{b_i - a_{ij}x_j}{a_{ik}}$$

- Disaggregation

Example (from Miplib-problem p0282)

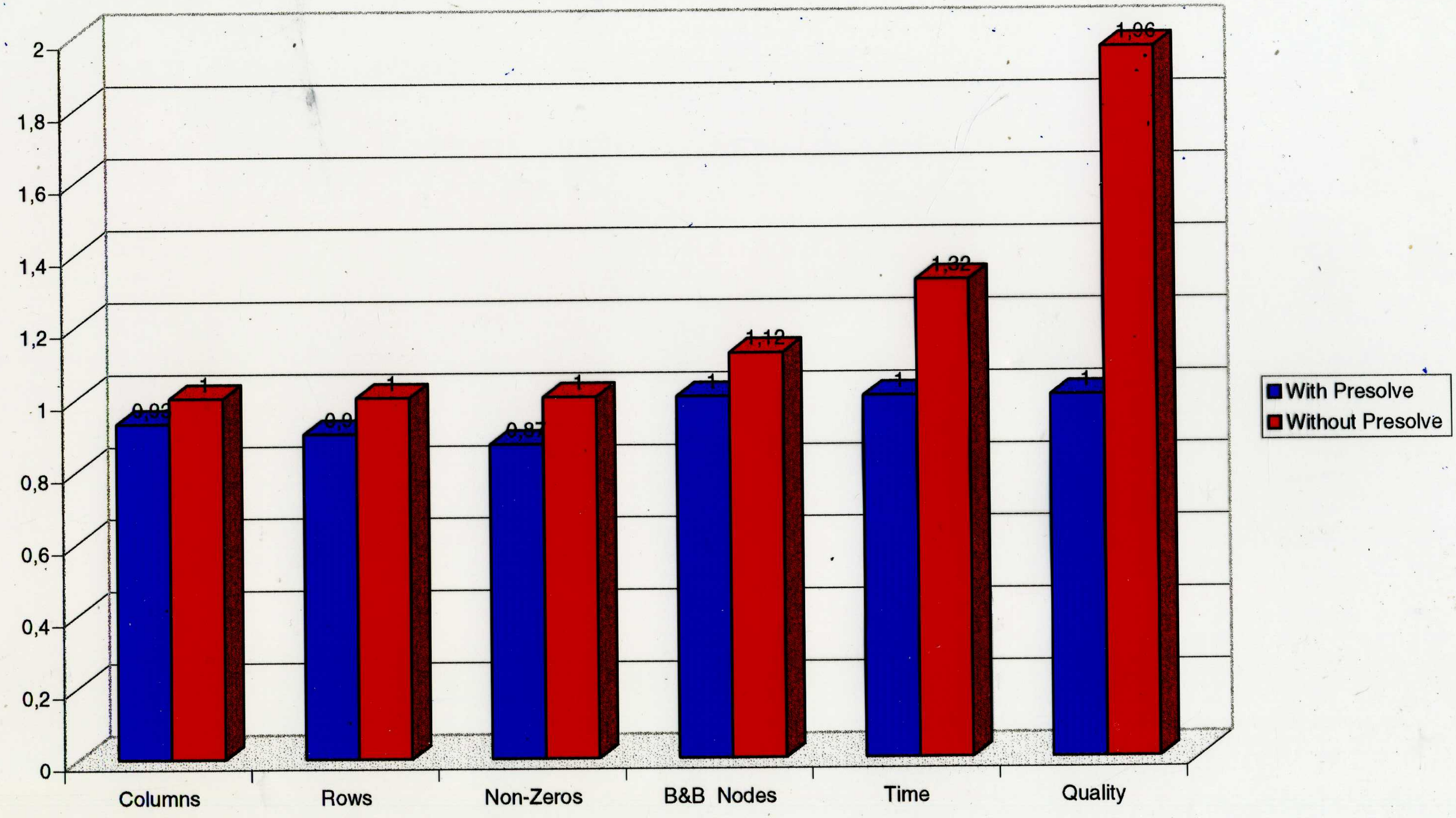
(2) $$x_{85} + x_{90} + x_{95} + x_{100} + x_{217} + x_{222} + x_{227} + x_{232} \leq 8x_{246},$$

where $x_i \in \{0, 1\}$. Replace (2) by

$$x_i \leq x_{246}$$

for $i \in \{85, 90, 95, 100, 217, 222, 227, 232\}$.

Presolve: A Comparison



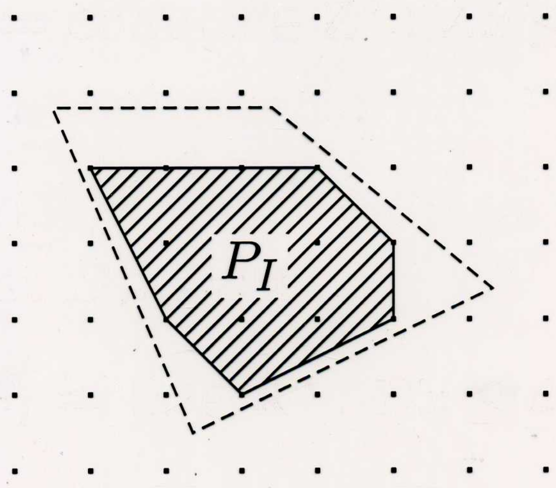
Cutting Planes

$$P_I = \text{conv}\{x \in \mathbb{Z}^n : Ax \leq b\}$$

is a polyhedron.

$\Longrightarrow$ There exist D, d with

$$P_I = \{x \in \mathbb{R}^n : Dx \leq d\}$$



How to find the right inequalities from D, d?

Schnittebenen:

Ungleichungen, die Struktur ausnützen

- 0/1 Rucksack

$$P_K = \text{conv}\{\, x \in \{0,1\}^n : \sum_{i=1}^{n} f_i x_i \le F \,\}$$

mit $f_i, F \in \mathbb{N}$.

- Set Covering / Packing

$$P_S = \text{conv}\{\, x \in \{0,1\}^n : Ax \begin{Bmatrix} \ge \\ = \\ \le \end{Bmatrix} \mathbb{1} \,\}$$

mit $A \in \{0,1\}^{m \times n}$.

Strukturunabhängige Ungleichungen

- Gomory Schnitte.
  Gomory'58, Gomory'60, Chvátal'73
- Lift-and-Project / Disjunktive Schnitte.
  Balas, Ceria, Cornuéjols'93
  Lovász, Schrijver'91
  Sherali, Adams'90
- Mixed Integer Rounding (MIR) Schnitte.

  Nemhauser, Wolsey'90
  Marchand, Wolsey'98

Weitere Ungleichungen

- Verallgemeinere Idee des Intersektionsgraphen

  Atamturk, Nemhauser, Savelsbergh'98

  Borndörfer, Weismantel'98

- Verallgemeinere den Rucksack auf
  - GUB Bedingungen

    Johnson, Padberg'81, Gu, Nemhauser, Savelsbergh'98
  - ganzzahlige Variablen

    Ceria, Cordier, Marchand, Wolsey'98
  - kontinuierliche Variablen

    Marchand, Wolsey'98, Marchand, Wolsey'99
  - mehr als eine Basisungleichung

    Martin, Weismantel'98

- Flow Cover Ungleichungen

  Padberg, Van Row, Wolsey'85

  Gu, Nemhauser, Savelsbergh'99

- Sequenz-unabhängiges Liften

  Wolsey'77

  Gu, Nemhauser, Savelsbergh'95

➡ Marchand, Martin, Weismantel, Wolsey: "Cutting Planes *in Integer and Mixed Integer Programming*", CORE Discussion Paper 9953, 1999.

Performance of Cuts

➥ Bixby, Fenelon, Gu, Rothberg, Wunderling:
"MIP: Theory and Practice, Closing the Gap",
Technical Report, 1999.

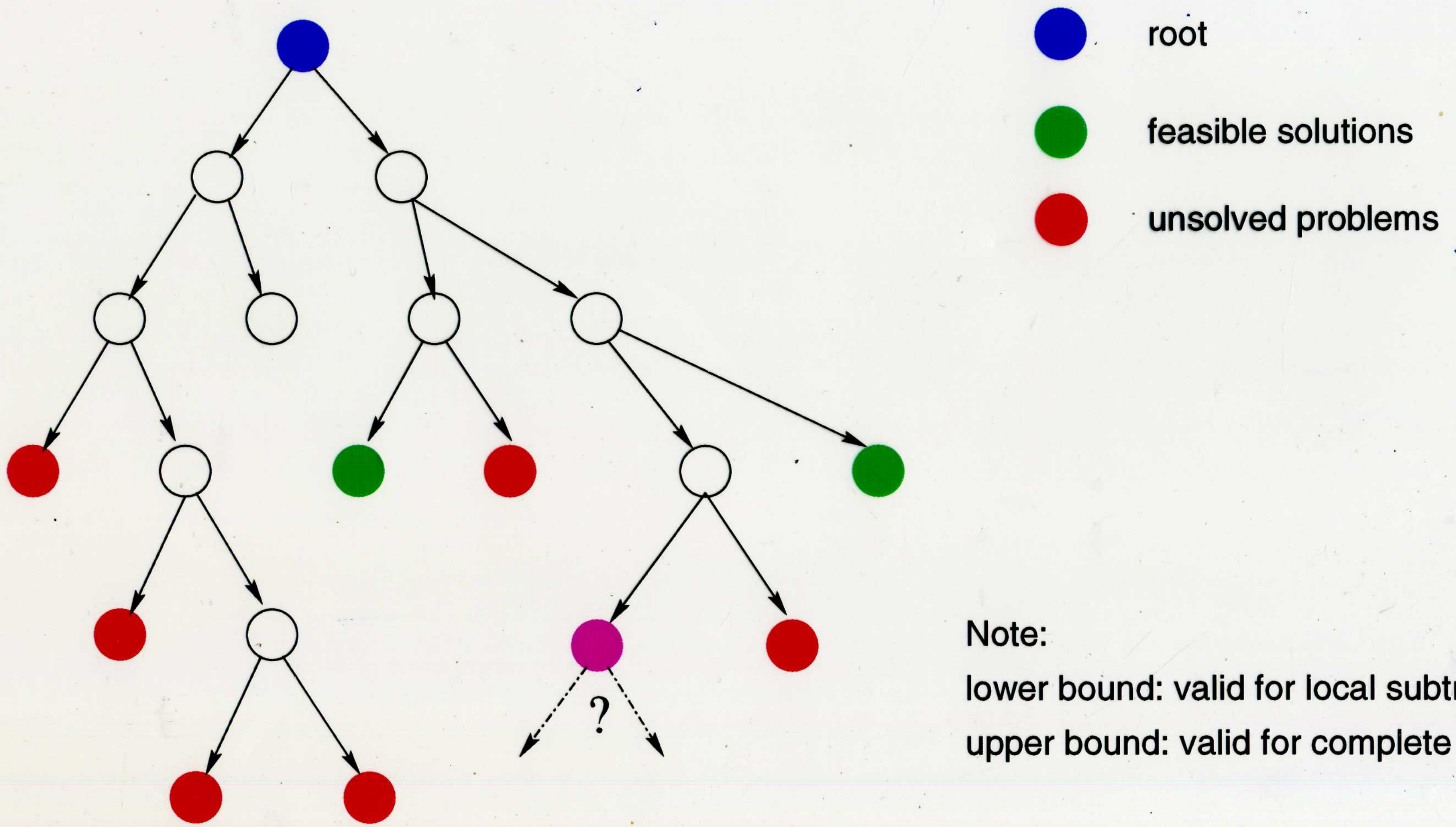
- Relative to Defaults

Cliques	0%
GUB covers	0%
Flow Covers	12%
Covers	16%
Gomory cuts	35%

- Individual Cuts

Cliques	0%
GUB covers	10%
Flow Covers	18%
Covers	58%
Gomory cuts	97%

Branch-and-Bound Tree



Note:

lower bound: valid for local subtree

upper bound: valid for complete tree

Node Selection:

(1) Depth-First-Search (DFS)

➥ Choose node "deepest" in the branch-and-bound tree

Goals:

- Process nodes fast
- Find feasible solutions
- Minimize memory requirements

(2) Best First Search (BFS)

➥ Choose node with worst lower bound

Goal:

- Improve global lower bound

(3) Best Estimate (Mitra'73, Hirst)

Evaluate nodes according to their potential of having a better feasible solution

➥ Choose node p that minimizes

$$\varrho(p) = z_{LP}(p) + \frac{z^*_{IP} - z_{LP}(\text{root})}{s(\text{root})} \cdot s(p),$$

where

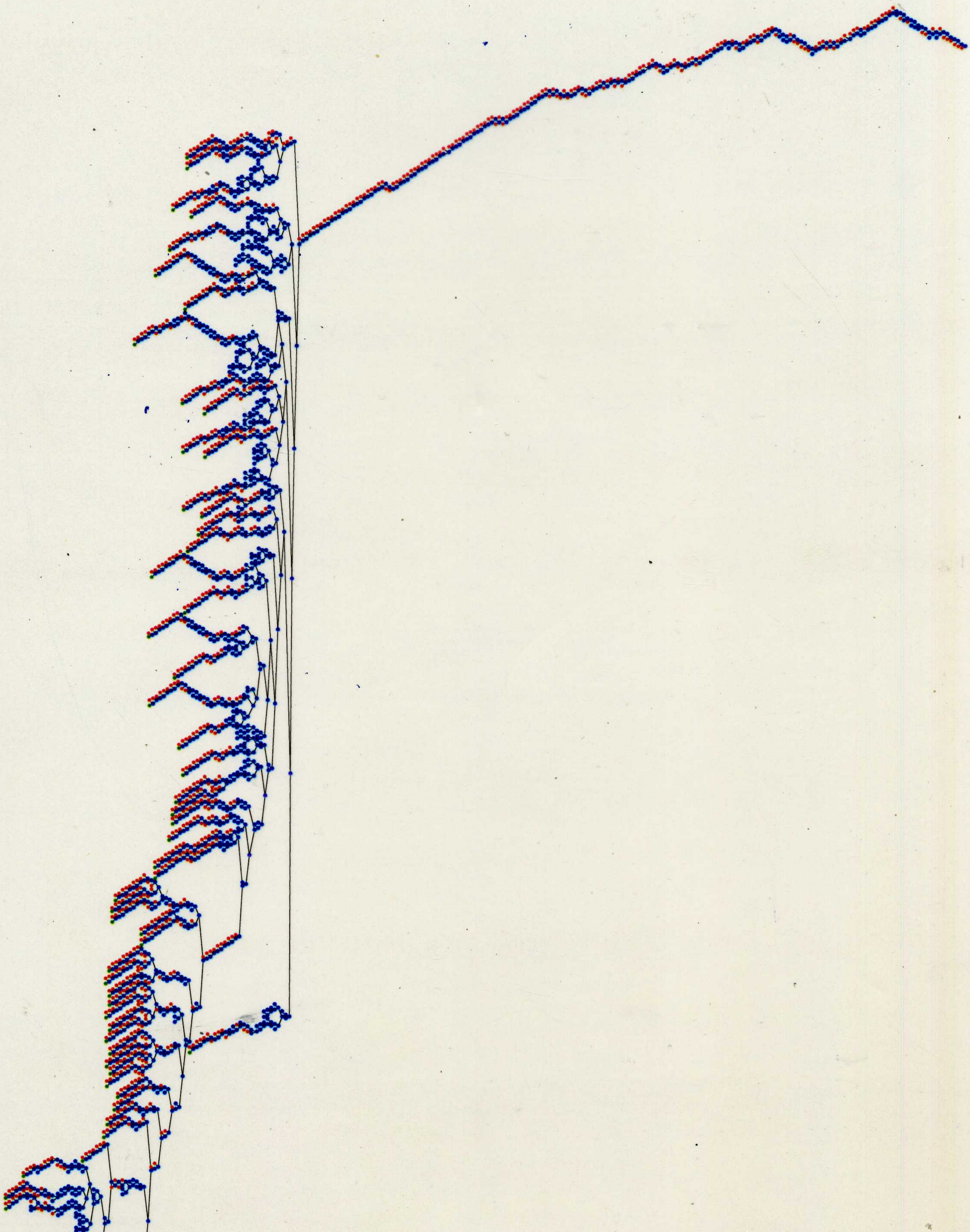
$$s(p) = \sum_i \min\{x^*_i - \lfloor x^*_i \rfloor, \lceil x^*_i \rceil - x^*_i\}$$

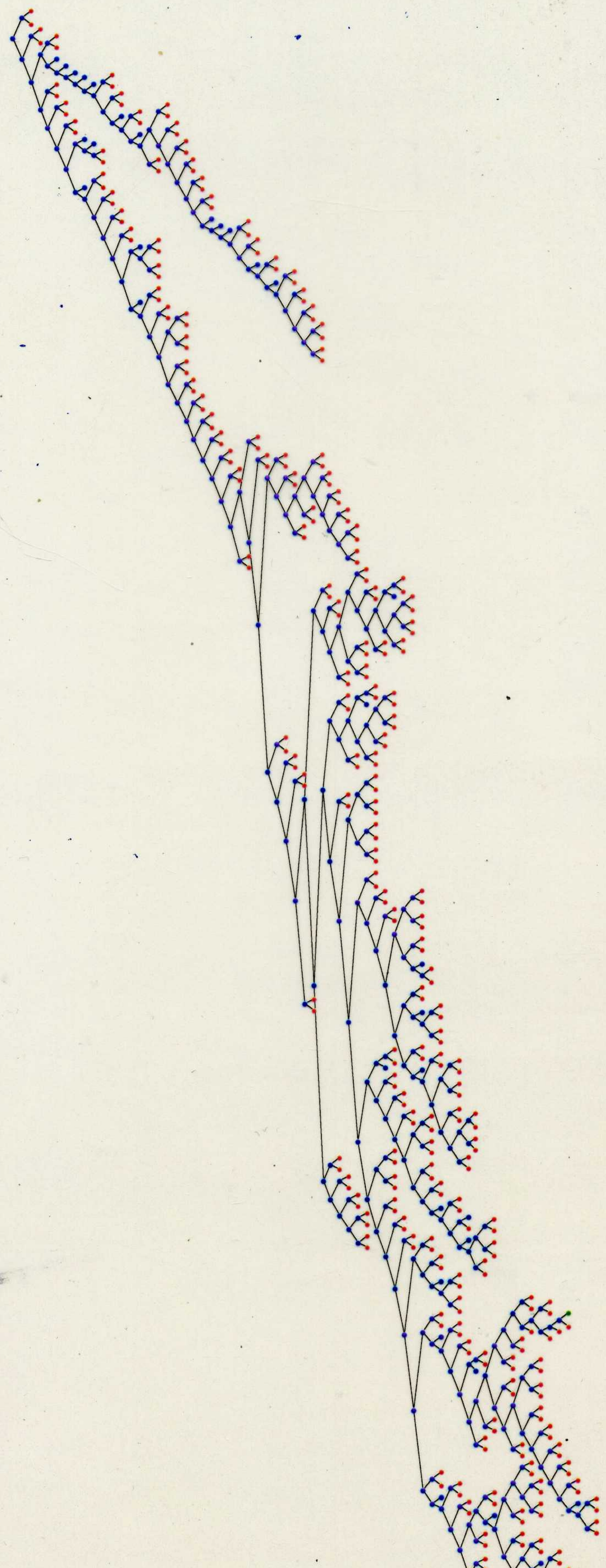
Goal:

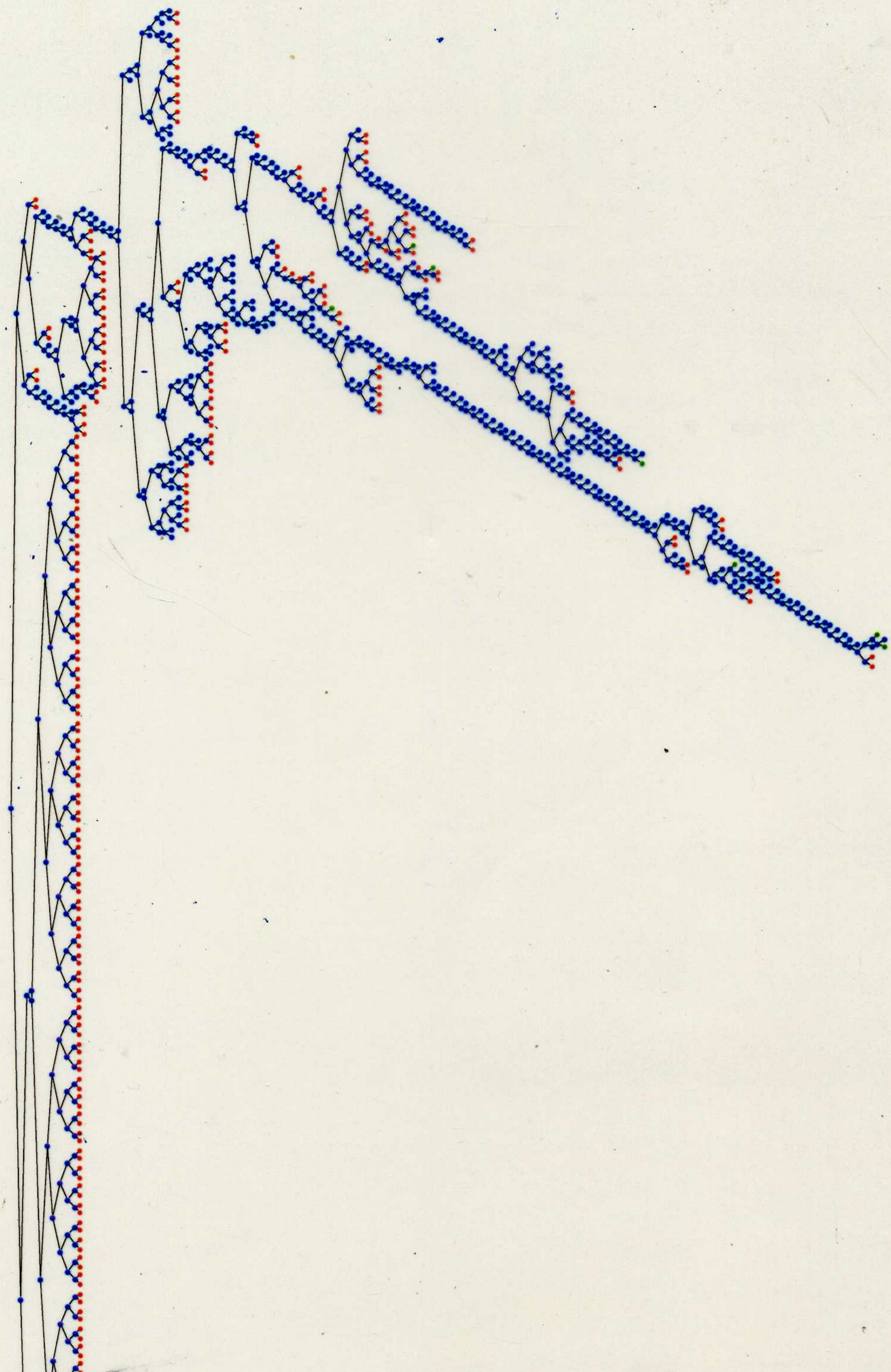
- Improve best feasible solution

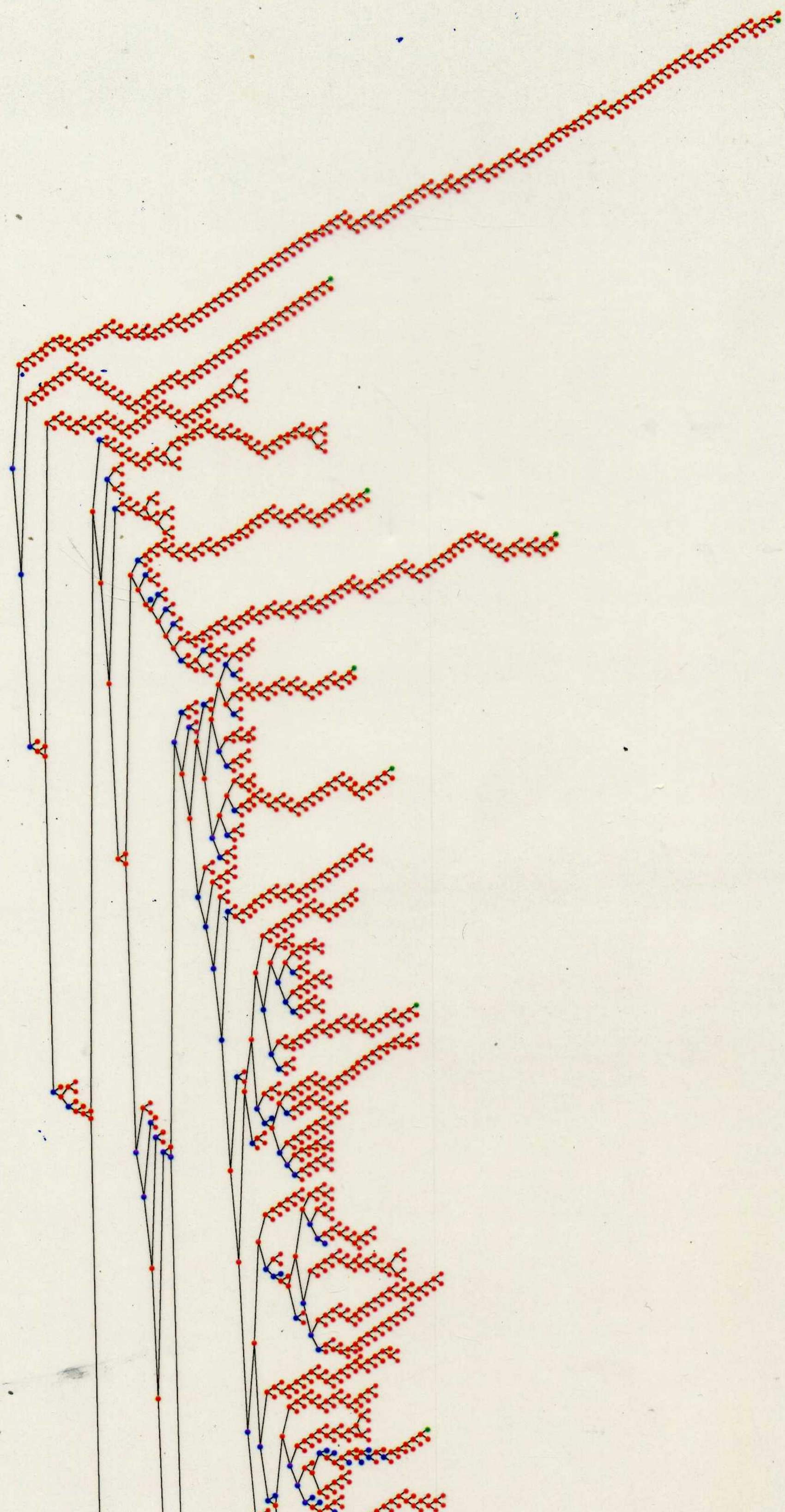
(4) Default in SIP

Mixture of (1), (2) and (3)

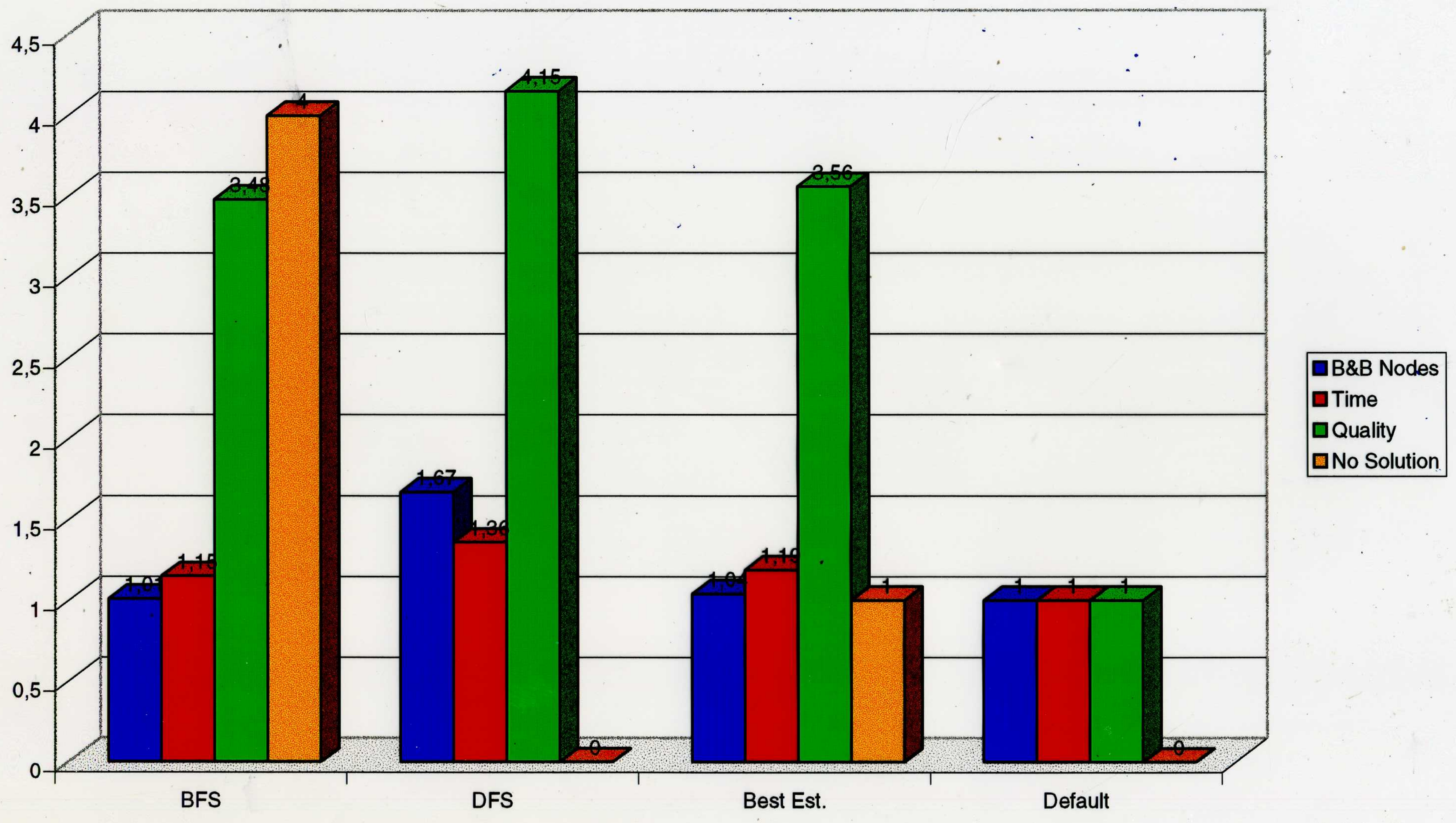








Node Selection: A Comparison



Node Selection: A Comparison