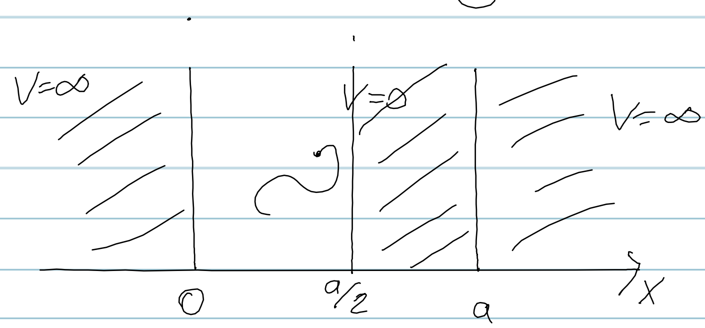
$p_{x,1}^2 = \frac{\hbar^2\pi^2}{a^2} = 2mE_1$ $\qquad E_{kin} = \frac{p^2}{2m}$

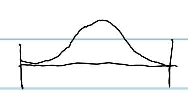
$p_{x,1} = \pm\sqrt{2mE_1}$

Genauigkeit: $2\sqrt{2mE_1}$

Vorlesung 22.11.2013



$E_n = \frac{n^2h^2}{8ma^2}$ $\quad n = 1, 2, 3$ $\qquad E_1 = \frac{h^2}{8ma^2}$

Aufenthaltswahrscheinlichkeit ψ_1^2 

Impuls $p_x = \pm\sqrt{2mE_1}$

$p_x = \pm\sqrt{2\cancel{m}\frac{h^2}{8\cancel{m}a^2}} = \pm\cancel{\sqrt{\;}}\ \frac{h}{2a}$

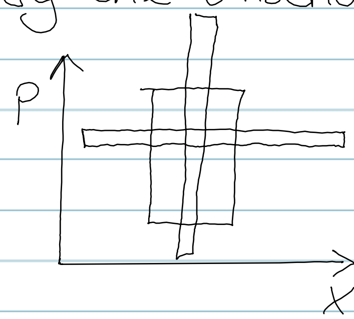
Unsicherheit $2\frac{h}{2a} = \frac{h}{a}$

$a/2$ " $\quad 2\frac{h}{2(\frac{a}{2})} = 2\frac{h}{a}$

$$\boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$$ Heisenberg'sche Unschärferelation

$$\Delta x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\Delta p \equiv \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$



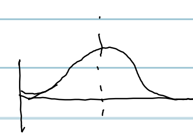
erfüllt für Teilchen im Kasten? ψ_1

$$\langle x\rangle = \int_0^a \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} \cancel{dx}\, x \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} dx$$

$$= \frac{2}{a}\int_0^a x \cdot \sin^2\frac{\pi x}{a} dx$$

Subst.: $\varphi = \frac{\pi x}{a} \rightarrow \frac{d\varphi}{dx} = \frac{\pi}{a}$

$$= \frac{2}{a}\frac{a}{\pi}\frac{a}{\pi}\underbrace{\int_0^\pi \varphi \sin^2\varphi\, d\varphi}_{\frac{\pi^2}{4}} = \frac{a}{2}$$



$$\langle x^2\rangle = \int_0^a \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}\, x^2 \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{2}{a}\int_0^a x^2 \sin^2\frac{\pi x}{a} dx = \frac{2}{a}\frac{a^3}{\pi^3}\int_0^\pi \varphi^2\sin^2\varphi\, d\varphi = \frac{a^2}{3} - \frac{a^2}{2\pi^2} = \langle x^2\rangle$$

$\langle p\rangle = 0$ s.o.

$\langle p^2\rangle = \frac{\hbar^2\pi^2}{a^2}$ s.o.

$$\Delta x \cdot \Delta p = \left(\frac{a^2}{3} - \frac{a^2}{2\pi^2} - \frac{a^2}{4}\right)^{1/2}\left(\frac{\hbar^2\pi^2}{a^2} - 0\right)^{1/2} = \left(\frac{a^2}{12} - \frac{a^2}{2\pi^2}\right)^{1/2}\frac{\hbar\pi}{a}$$

$$= \hbar\pi\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)^{1/2} = \hbar\left(\frac{\pi^2}{12} - \frac{1}{2}\right)^{1/2} = \hbar\underbrace{\left(\frac{\pi^2-6}{12}\right)^{1/2}}_{>\frac{1}{2}} \geq \frac{\hbar}{2}$$

c) Teilchen im 3-dim. Kasten

Anwendung: Translationszustandssumme
$\rightarrow$ PC III

Eigenwertgl. $\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi}$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2} = -\frac{2mE}{\hbar^2}\psi$$

Lösungsansatz: $\psi = X(x)\,Y(y)\,Z(z) = XYZ$

$$YZ\frac{\partial^2 X}{\partial x^2}+XZ\frac{\partial^2 Y}{\partial y^2}+XY\frac{\partial^2 Z}{\partial z^2} = -\frac{2mE}{\hbar^2}XYZ \quad \Big|: XYZ$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\frac{2mE}{\hbar^2}$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{2mE}{\hbar^2}}_{c} = \underbrace{-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{c}$$

$$\boxed{-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \frac{2mE_z}{\hbar^2}} = c$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{2m(E-E_z)}{\hbar^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{2m(E-E_z)}{\hbar^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

$$\boxed{-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{2mE_y}{\hbar^2}} \longrightarrow \boxed{-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{2mE_x}{\hbar^2}}$$

Produktansatz von WF

$$\boxed{\psi = XYZ}$$

$$\psi = \sqrt{\frac{8}{abc}} \sin \frac{n_x \pi}{a} x \sin \frac{n_y \pi}{b} y \sin \frac{n_z \pi}{c} z \qquad n_x, n_y, n_z = 1, 2, 3, \ldots$$

Energieeigenwerte

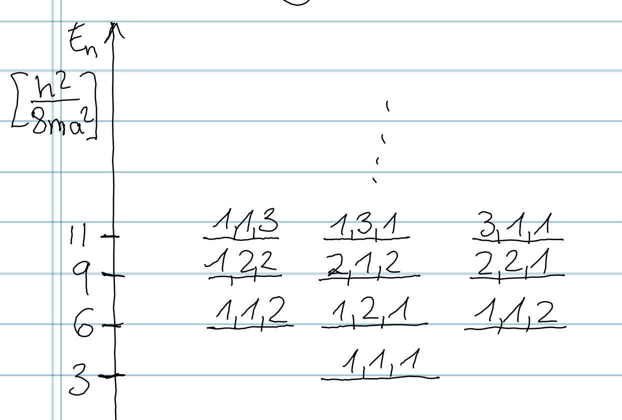
$$\boxed{E = E_x + E_y + E_z}$$

$$\hookrightarrow \boxed{E = \frac{h^2}{8m} \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right)}$$

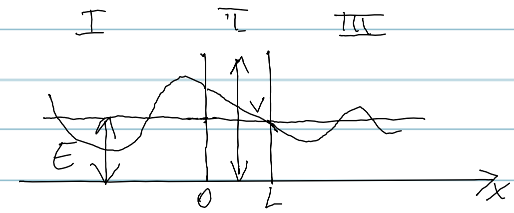
Würfel: $a = b = c$

$$E = \frac{h^2}{8ma^2} (n_x^2 + n_y^2 + n_z^2)$$

Entartung: $E(1,1,2) = E(1,2,1) = E(2,1,1) = \frac{6h^2}{8ma^2}$



d) Tunneleffekt



Pot. ist endlich, $E < V$ } tunneln möglich

Bereich I: $V=0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$\psi = Ce^{ikx} + De^{-ikx} \qquad +k^2 = \frac{2mE}{\hbar^2}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$k = \sqrt{\frac{2m h^2}{\hbar^2 8ma^2}}$$

$$k = \sqrt{\frac{\cancel{2m} p^2}{\cancel{2m} \hbar^2}}$$

$$\rightarrow \boxed{k = \frac{p}{\hbar}}$$

Bereich II: $E < V$

$$\frac{d^2\psi}{dx^2} = \frac{2m\overset{(V-E)}{\cancel{(E-V)}}}{\hbar^2}\psi$$

$$\psi = Ee^{\kappa x} + Fe^{-\kappa x} \qquad \kappa = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

Bereich III : $V=0$

$$\psi = C' e^{ikx} + D' e^{-ikx}$$

$\rightarrow$ Randbed. $\rightarrow$ $\rightarrow$ Koeff. (Annahme : $V \gg E$)
$\kappa \cdot a \gg 1$

Ziel: Transmissionswahrscheinlichkeit T
Tunnel — " —

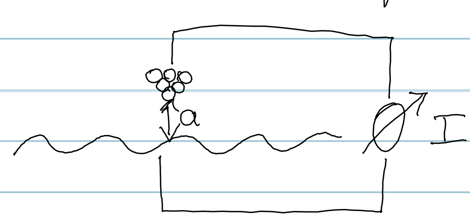
$$T = \frac{|C'|^2}{|C|^2}$$

$$\boxed{T = \frac{16E(V-E)}{V^2} e^{-2\kappa \cdot a}} \qquad \kappa = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

$T \uparrow$ $m \downarrow$ $(V-E) \downarrow$

2.5 Anwendungen

Rastertunnelmikroskopie



Tunnelstrom $I \propto T$

Bsp: $\left.\begin{array}{l} a_1 = 0{,}5\,\text{nm} \\ a_2 = 0{,}6\,\text{nm} \end{array}\right\}$ $\frac{I(a_2)}{I(a_1)} = \frac{T(a_2)}{T(a_1)} = e^{-2\kappa(a_2 - a_1)} = 0{,}23$