Wiederholung:

$\hat{H}\Psi_n = E_n \cdot \Psi_n$

Eigenfunktion   Eigenwert
$\hookrightarrow E_n \in \mathbb{R}$

$\hat{H}$ = hermitisch

$\int f^* \hat{H} g \, d\tau = \int (\hat{H} f)^* \cdot g \, d\tau$

$\Leftrightarrow \langle f | \hat{H} g \rangle = \langle \hat{H} f | g \rangle$

(müssen aber quadrat-integrierbar sein)

$\langle \Psi_n | \Psi_m \rangle = \delta_{nm}$: vollständig

$f = \sum_n^\infty c_n \cdot \Psi_n$

$\rightarrow c_n$ über Projektion $c_n = \langle \Psi_n | f \rangle$

wenn f normiert wird

$\hookrightarrow \underbrace{\langle f | f \rangle} = 1 = \sum_n c_n^* c_n = \sum_n |c_n|^2 = 1$ $|c_n|^2 \geq 0 \Rightarrow P_n = |c_n|^2$

$|f|^2 \geq 0$

$\hookrightarrow \int f^* \cdot f \, d\tau = \int \underbrace{|f|^2}_{\text{Wahrscheinlichkeitsdichte}} d\tau = 1$

$\rightarrow$ Wahrscheinlichkeit bestimmen: $\iiint_a^b \underbrace{f(x,y,z)}_{|f|^2} d\tau = P(a \leq x,y,z \leq b)$

$\int f^* \hat{H} f \, d\tau = \int \left(\sum_n c_n^* \Psi_n^*\right) \hat{H} \left(\sum_m c_m \Psi_m\right) d\tau$

$\langle f | \hat{H} f \rangle = \sum_n \sum_m c_n^* c_m \langle \Psi_n | \underbrace{\hat{H} \Psi_m}_{= E_m \Psi_m} \rangle$

$= \sum_n \sum_m c_n^* c_m \cdot E_m \cdot \delta_{nm}$

$\langle f | \hat{H} f \rangle = \sum_n |c_n|^2 \cdot E_n = \sum_n P_n \cdot E_n = \langle E \rangle$ Erwartungswert

$\langle A \rangle = \langle f | \hat{A} f \rangle$

falls f nicht normiert ($\langle f | f \rangle \neq 1$): $\langle A \rangle = \frac{\langle f | \hat{A} f \rangle}{\langle f | f \rangle}$

falls $c_n = \delta_{nk} \Rightarrow f = \Psi_k \Rightarrow \hat{A} f_k = \lambda_k f_k \Rightarrow \langle A \rangle = \lambda_k$

Physikalische Chemie II   Mathesemínar II   15.11.18

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$|f|^2 > 0$

$\hookrightarrow \int f^* \cdot f \, d\tau = \int \underbrace{|f|^2}_{\text{Wahrscheinlichkeitsdichte}} d\tau = 1$

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$\int f^* \hat{H} f \, d\tau = \int \left(\sum_n c_n^* \Psi_n^*\right) \hat{H} \left(\sum_m c_m \Psi_m\right) d\tau$

$\langle f | \hat{H} f \rangle = \sum_n \sum_m c_n^* c_m \langle \Psi_n | \underbrace{\hat{H} \Psi_m}_{= E_m \cdot \Psi_m} \rangle$

$= \sum_n \sum_m c_n^* c_m \cdot E_m \cdot \delta_{nm}$

$\langle f | \hat{H} f \rangle = \sum_n |c_n|^2 \cdot E_n = \sum_n P_n \cdot E_n = \langle E \rangle$ $\rightarrow$ Erwartungswert

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falls $c_n = \delta_{nk} \Rightarrow f = \Psi_k \Rightarrow \hat{A} f_k = \lambda_k f_k \Rightarrow \langle A \rangle = \lambda_k$

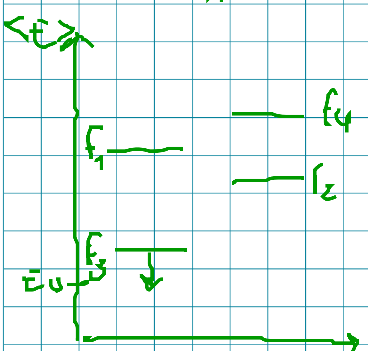
Es soll gelten $E_0 < E_1 < E_2 < \dots$

$$\langle E\rangle = \sum_n |c_n|^2 E_n = |c_0|^2 \cdot E_0 + \sum_{n=1}^{\infty} |c_n|^2 E_n$$

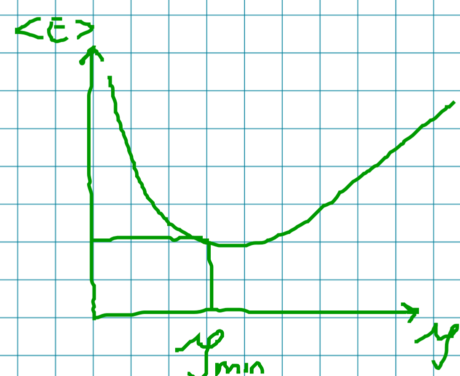
$$\sum_{n=0}^{\infty} |c_n|^2 = 1 \Rightarrow |c_0|^2 = 1 - \sum_{n=1}^{\infty} |c_n|^2$$

$$\langle E\rangle = \left(1 - \sum_{n=1}^{\infty} |c_n|^2\right) \cdot E_0 + \sum_{n=1}^{\infty} |c_n|^2 E_n$$

$$= E_0 + \sum_{n=1} |c_n|^2 (E_n - E_0) \geq E_0$$ ⇝ Grundzustandsenergie



$$f(\varphi) \to \langle A\rangle(\varphi) = \langle f|\hat{A}|f\rangle$$



$$\frac{d\langle E\rangle(\varphi)}{d\varphi} = 0$$

↳ $\langle E\rangle(\varphi_{min}) \geq E_0$

Variationsverfahren

$$f = \sum_n c_n \varphi_n \to \langle E\rangle = \left\langle \sum_n c_n \varphi_n \middle| \hat{H} \sum_m c_m \varphi_m \right\rangle$$

$$= \sum_n \sum_m c_n^* c_m \underbrace{\langle \varphi_m | \hat{H} \varphi_n \rangle}_{H_{mn}}$$

$$= \sum_n \sum_m c_n^* c_m H_{nm}$$

$$\hat{H} \approx \begin{pmatrix} H_{11} & H_{12} & \dots & H_{1\,10} \\ \vdots & & & \\ H_{10\,1} & \dots & & H_{10\,10} \end{pmatrix} \quad H_{nm} = H_{mn}^* \;;$$ hermitische Matrix

$$\frac{\partial \langle E\rangle}{\partial c_n} = 0 \to 10 \times 10$$ lineares GS, homogen   Ritzsches Variationsverfahren

$$\hat A \hat B \neq \hat B \hat A \Rightarrow [\hat A, \hat B]_- = \hat A \hat B - \hat B \hat A$$

$$[\hat A, \hat B]_- f = 0 \rightarrow \hat A f = \lambda f$$

$$\hat B f = \mu f$$

$$\langle \underset{\hat B}{\hat A} \psi | \psi \rangle = \langle \psi | \underset{\hat B}{\hat A} \psi \rangle = \langle A \rangle$$

$$[\hat A, \hat B]_- \neq 0$$

$$\Delta \hat A = \hat A - \langle A \rangle \qquad \Delta \hat B = \hat B - \langle B \rangle$$

$$\sigma_{\underset{B}{A}}^2 = \langle \Delta \underset{B}{A}^2 \rangle = \langle \psi | \Delta \underset{B}{\hat A}^2 \psi \rangle = \langle \Delta \hat A \psi | \Delta \hat A \psi \rangle = \| \langle \Delta \hat A \psi \rangle \|^2$$

$$\sigma_B^2 = \| \langle \Delta \hat B \psi \rangle \|^2$$

$$\sigma_A^2 \cdot \sigma_B^2 = \| \langle \Delta \hat A \psi \rangle \|^2 \cdot \| \langle \Delta \hat B \psi \rangle \|^2 \geq | \langle \Delta \hat A \psi | \Delta \hat B \psi \rangle |^2 = | \langle \psi | \underbrace{\Delta \hat A \Delta \hat B}_{*} \psi \rangle |^2$$

Cauchy-Schwarz-Ungleichung

$$\langle f | f \rangle \cdot \langle g | g \rangle \geq | \langle f | g \rangle |^2 \Rightarrow \| f \|^2 \cdot \| g \|^2 \geq | \langle f | g \rangle |^2$$

$$* \; (\hat A - \langle A \rangle)(\hat B - \langle B \rangle) = \hat A \hat B - \hat A \langle B \rangle - \langle A \rangle B + \langle A \rangle \langle B \rangle$$

$$\langle \Delta \hat A \psi | \Delta \hat B \psi \rangle = \langle \psi | \hat A \hat B \psi \rangle - \langle B \rangle \underbrace{\langle \psi | \hat A \psi \rangle}_{\langle A \rangle} - \langle A \rangle \underbrace{\langle \psi | \hat B \psi \rangle}_{\langle B \rangle} + \langle A \rangle \langle B \rangle \underbrace{\langle \psi | \psi \rangle}_{1}$$

$$= \langle \psi | \hat A \hat B \psi \rangle - \langle A \rangle \langle B \rangle$$

voneinander abziehen

$$\langle \Delta \hat B \psi | \Delta \hat A \psi \rangle = \langle \psi | \hat B \hat A \psi \rangle - \langle B \rangle \langle A \rangle$$

$$\langle \Delta \hat A \psi | \Delta \hat B \psi \rangle - \langle \Delta \hat B \psi | \Delta \hat A \psi \rangle = \langle \psi | [A B]_- \psi \rangle = 2i \, \mathrm{Im}(\langle \Delta \hat A \psi | \Delta \hat B \psi \rangle)$$

$$\Downarrow$$

$$\langle \Delta \hat A \psi | \Delta \hat B \psi \rangle$$

$$\langle f | g \rangle = \int f^* g \, d\tau$$

$$\langle g | f \rangle = \int g^* f \, d\tau$$
$$= \left(\int g \cdot f^* d\tau \right)^*$$
$$= \langle f | g \rangle^*$$

$z = a + ib$
$z^* = a - ib$
$z - z^* = 2ib$
$|z|^2 = a^2 + b^2$

$$|\langle \Delta\hat{A}\psi | \Delta\hat{B}\psi \rangle|^2 = Re^2(\langle \ \rangle) + Im^2(\langle \ \rangle)$$
$$\sigma_A^2 \sigma_B^2 \geq Im^2(\langle \ \rangle)$$
$$= \left(\frac{\langle \psi | [A,B]_- \psi}{2i} \right)^2$$

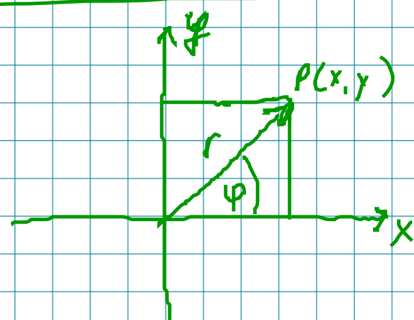
$$\boxed{\sigma_A \cdot \sigma_B \geq \frac{1}{2i} |\langle \psi | [\hat{A},\hat{B}]_- \psi \rangle|} \Rightarrow \Delta x \Delta p \geq \frac{1}{2i} i\hbar = \frac{\hbar}{2}$$

$\hat{A} = x \qquad \hat{B} = -i\hbar \frac{d}{dx}$

$$[\hat{A},\hat{B}]_- = \left[x, -i\hbar \frac{d}{dx}\right]_- = -i\hbar \left(x \frac{d}{dx} - \frac{d}{dx}(x) \right)$$

$$= -i\hbar$$

Polarkoordinaten



$x = r \cdot \cos\varphi$
$y = r \sin\varphi$

$r = \sqrt{x^2 + y^2}$
$\varphi = \arctan\left(\frac{y}{x}\right)$

$$f(x,y) = f(x[r,\varphi], y[r,\varphi]) = \tilde{f}(r,\varphi)$$

Vorteil: $\tilde{f}(r,\varphi) = \tilde{f}_r(r) \cdot \tilde{f}_\varphi(\varphi)$
$\tilde{f}(r,\varphi) = \tilde{f}(r)$

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$$dx = \left(\frac{\partial x}{\partial r}\right)_\varphi dr + \left(\frac{\partial x}{\partial \varphi}\right)_r d\varphi$$

$$dx = \cos\varphi\, dr - (r\cdot\sin\varphi)\, d\varphi$$

$$dy = \sin\varphi\, dr + r\cdot\cos\varphi\, d\varphi$$

$$= f_x(\cos\varphi\, dr - r\sin\varphi\, d\varphi)$$

$$+ f_y(\sin\varphi\, dr + r\cos\varphi\, d\varphi)$$

$$= \overbrace{(\cos\varphi\, f_x + \sin\varphi\, f_y)}^{\Rightarrow \frac{\partial f}{\partial r}} dr$$

$$+ \underbrace{(-r\sin\varphi\, f_x + r\cdot\cos\varphi\, f_y)}_{\frac{\partial f}{\partial \varphi}} d\varphi = d\tilde{f}$$

$$\tilde{f}(r,\varphi) = \tilde{f}(r) \qquad \frac{\partial \tilde{f}}{\partial r} = \frac{d\tilde{f}}{dr} = (f_x, f_y)\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix}$$

Nabla-Operator: $\vec{\nabla} = \begin{pmatrix}\partial/\partial x \\ \partial/\partial y\end{pmatrix}$ $\qquad \Delta = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Laplace-Operator.

Physikalische Chemie II Matheseminar 16.11.2018

Wiederholung:

Polarkoordinaten

$(x, y) \longrightarrow (r, \varphi)$ $\qquad x = r\cdot\cos\varphi \qquad y = r\sin\varphi$

$r = +\sqrt{x^2+y^2} \qquad \varphi = \arctan\left(\frac{y}{x}\right)$

$-\infty < x, y < \infty$

$0 \leq \varphi \leq 2\pi$

$0 \leq r \leq \infty$

$f(x,y) \longrightarrow \tilde{f}(r,\varphi)$

$df = d\tilde{f}$

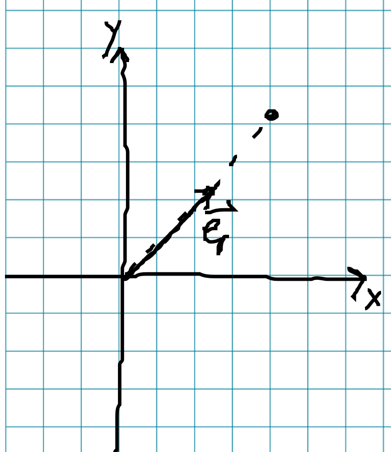
$df = f_x\, dx + f_y\, dy \qquad dx = \cos\varphi\, dr + (-1 \sin\varphi)\, d\varphi$

$d\tilde{f} = f_r\, dr + f_\varphi\, d\varphi \qquad dr = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\, dx + \frac{y}{\sqrt{y^2+x^2}}\, dy$

$\vec{\nabla} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} \qquad \Delta = \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\tilde{f}(r,\varphi) = \tilde{f}(r)$

$$\vec{\nabla}\tilde{f} = \begin{pmatrix} \partial\tilde{f}/\partial x \\ \partial\tilde{f}/\partial y \end{pmatrix} = \begin{pmatrix} \frac{d\tilde{f}}{dr} \cdot \frac{x}{r} \\ \frac{d\tilde{f}}{dr} \cdot \frac{y}{r} \end{pmatrix} = \frac{1}{r} \cdot \frac{d\tilde{f}}{dr} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \frac{d\tilde{f}}{dr} \cdot \underbrace{\frac{\vec{r}}{r}}_{\vec{e}_r}$$

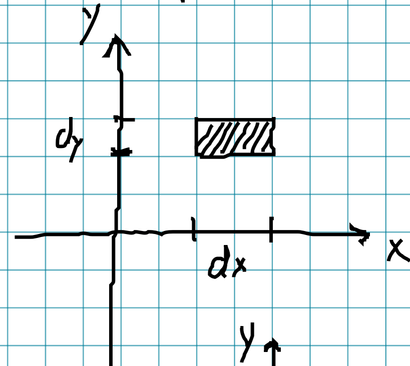


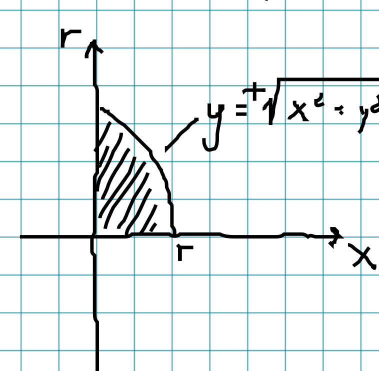
$$\frac{\partial}{\partial x} = \frac{\partial\tilde{f}}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial\tilde{f}}{\partial\varphi} \cdot \frac{\partial\varphi}{\partial x}$$

$$\frac{\partial}{\partial y} = \frac{\partial\tilde{f}}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial\tilde{f}}{\partial\varphi} \cdot \frac{\partial\varphi}{\partial y}$$

$$\frac{\partial^2}{\partial x^2} = \Big(\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{r}} \cdot \frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial x}\, \frac{\partial}{\partial\varphi}\Big) \cdot \Big(\underbrace{\frac{\partial r}{\partial x}}_{\frac{x}{R}} \cdot \frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial x} \cdot \frac{\partial}{\partial\varphi}\Big)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \underbrace{dx\, dy}_{\text{Flächenelement}}$$



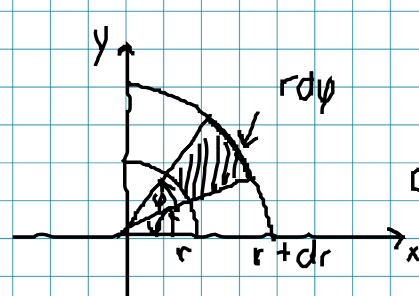


$f(x,y) = f_1(x) \cdot f_2(y)$

(1) $f \longrightarrow \tilde{f}(r,\varphi)$

(2) Grenzen $x\, y \longrightarrow$ Grenzen von $r\,\varphi$

(3) $dx\,dy \longrightarrow dr\,d\varphi$



$d\tau = r\, dr\, d\varphi$
(merken)

$$\int_0^{2\pi}\int_0^{\infty} \tilde{f}(r,\varphi)\cdot r\, dr\, d\varphi \qquad (1)\ \tilde{f}(r,\varphi) = R(r)\cdot\Phi(\varphi)$$

$$\rightarrow \left(\int_0^{\infty} r\cdot R(r)\,dr\right)\cdot\left(\int_0^{2\pi}\Phi(\varphi)\,d\varphi\right)$$

$$(2)\ \tilde{f}(r,\varphi) = R(r) \qquad \Phi(\varphi) = 1$$

$$\int_0^{2\pi} 1\, d\varphi = 2\pi$$

$$R(r) = 1 \qquad \int_0^{R_0} 2\pi r\, dr = \pi R_0^2 \qquad R(r) = \delta(r - r_0) \text{ Deltafunktion}$$

$$R(r) = \delta(r - r_0) \qquad \int_0^{\infty} 2\pi r\cdot\delta(r - r_0)\,dr = 2\pi r_0 \qquad \int_{-\infty}^{\infty}\delta(r_0 - r)\cdot F(r)\,dr = F(r_0)$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx \qquad I_x = I_y = \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}$$

$$I^2 = I_x I_y = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \iint_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r\,dr\,d\varphi$$

$$= 2\pi\cdot\int_0^{\infty} r\, e^{-r^2}dr = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \pi$$

$$\left.\begin{aligned} dx &= \left(\frac{\partial x}{\partial r}\right)dr + \left(\frac{\partial x}{\partial \varphi}\right)d\varphi \\ dy &= \left(\frac{\partial y}{\partial r}\right)dr + \left(\frac{\partial y}{\partial \varphi}\right)d\varphi \end{aligned}\right\} \begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}}_{\text{Jacobimatrix} = \hat{J}}\cdot\begin{pmatrix} dr \\ d\varphi \end{pmatrix}$$

$$d\vec{r} = \hat{J}\cdot d\vec{q}$$

$$\hat{J}^{-1}d\vec{r} = \underbrace{\hat{J}^{-1}\cdot\hat{J}}_{\hat{E}}\, d\vec{q} = d\vec{q}$$

$$\begin{pmatrix} dx & 0 \\ 0 & dy \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix} = d\vec{r}$$

$$\hat{A} \cdot \vec{e} = d\vec{r}$$

$$\begin{pmatrix} dr & 0 \\ 0 & d\varphi \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} dr \\ d\varphi \end{pmatrix}$$

$$\hat{Q} \cdot \vec{e} = d\vec{q}$$

$$\hat{A}\,\vec{e} = \hat{J} \cdot \hat{Q} \cdot \vec{e}$$

$$|\hat{A}| \cdot \cancel{|\vec{e}|} = |\hat{J}| \cdot |\hat{Q}| \cdot \cancel{|\vec{e}|}$$

$$|\hat{J}| = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

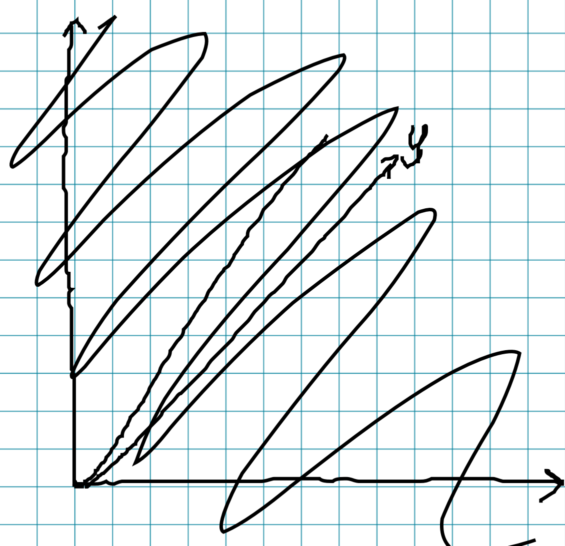
$$\left.\begin{array}{l} |\hat{A}| = dx\,dy \\ |\hat{Q}| = dr\,d\varphi \end{array}\right\} dx\,dy = |\hat{J}| \cdot dr\,d\varphi \;\Rightarrow\; dx\,dy = r\,dr\,d\varphi$$

Jacobi-Determinante

Kugelkoordinaten

$$(x, y, z) \longrightarrow (r, \varphi, \vartheta)$$

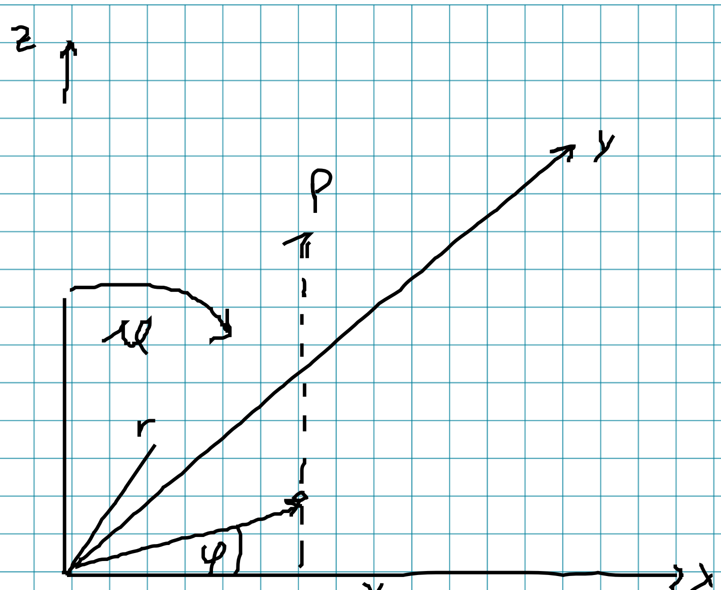
$$r = \sqrt{x^2 + y^2 + z^2}$$



ϑ = Breitengrad

$0 \leq \vartheta \leq \pi$

$0 \leq \varphi \leq 2\pi$



$\tan\varphi = \frac{y}{x}$

$x = r \cdot \sin\vartheta \cos\varphi$

$y = r \sin\vartheta \cdot \sin\varphi$

$z = r \cdot \cos\vartheta$

$x^2 + y^2 = r^2 \sin^2\vartheta$

$z^2 = r^2 \cos^2\vartheta$

$\tan^2\vartheta = \frac{x^2+y^2}{z^2}$

$\tan\vartheta = \sqrt{\frac{x^2+y^2}{z^2}}$

$f(x,y,z) = \tilde{f}(r,\vartheta,\varphi)$

$= R(r) \cdot Y(\vartheta,\varphi)$ → Kugelflächenfunktion ~~Kugelkoordinaten~~

$\downarrow$

$\Phi(\varphi) \cdot \Theta(\cos\vartheta)$

$\left(\frac{\partial x}{\partial r}\right) = \sin\vartheta \cos\varphi \qquad \left(\frac{\partial x}{\partial \vartheta}\right) = r \cdot \cos\vartheta \cos\varphi \qquad \left(\frac{\partial x}{\partial \varphi}\right) = -r \sin\vartheta \cdot \sin\varphi$

$\left(\frac{\partial y}{\partial r}\right) = \sin\vartheta \sin\varphi \qquad \left(\frac{\partial y}{\partial \vartheta}\right) = r \cos\vartheta \cdot \sin\varphi \qquad \left(\frac{\partial y}{\partial \varphi}\right) = r \sin\vartheta \cos\varphi$

$\left(\frac{\partial z}{\partial r}\right) = \cos\vartheta \qquad \left(\frac{\partial z}{\partial \vartheta}\right) = -r \sin\vartheta \qquad \left(\frac{\partial z}{\partial \varphi}\right) = 0$

$dx\,dy\,dz = |\hat{J}|\, dr\, d\vartheta\, d\varphi = r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$

$\tilde{f} = R(r)$

$$\iiint_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \tilde{f}(r,\vartheta,\varphi)\, r^2 \cdot \sin\vartheta\, dr\, d\vartheta\, d\varphi$$

$$= \left(\int_0^{\infty} r^2 \cdot R(r)\,dr\right)\left(\int_0^{\pi} \sin\vartheta\, d\vartheta\right) \cdot \left(\int_0^{2\pi} d\varphi\right)$$

Raumwinkel 4π: $= 2\pi \cdot \left[-\cos\vartheta\right]_0^{\pi} \cdot \int_0^{\infty} r^2 \cdot R(r)\,dr = 4\pi \int_0^{\infty} r^2 R(r)\,dr$

$$f_x(u_x) = N \cdot e^{\frac{-mu_x^2}{2k_BT}} \qquad \int_{-\infty}^{\infty} f(u_x)\,dx = 1$$

$$f(u_x, u_y, u_z) = f_x \; f_y \cdot f_z \qquad \iiint_{-\infty}^{\infty} f(u_x\, u_y\, u_z)\,du_x\,du_z\,du_y = 1$$

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

$$g(u, \vartheta, \varphi) = N^3 e^{-\frac{m \cdot u^2}{2kT}} \qquad \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} g(u, \vartheta, \varphi)\, u^2 \cdot \sin\vartheta\, du\, d\vartheta\, d\varphi$$

$$N^3 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-\frac{mv^2}{2kT}} \cdot u^2\, du\, d\vartheta\, d\varphi\, \sin\vartheta = 4\pi N^3 \int^{\infty} u^2 \cdot e^{-\frac{mu^2}{2kT}} du$$

$$= 1$$

$$\left.f(u) = 4\pi u^2 \; N^3 \cdot e^{-\frac{mv^2}{2kT}}\right\} \text{radiale Dichte}$$