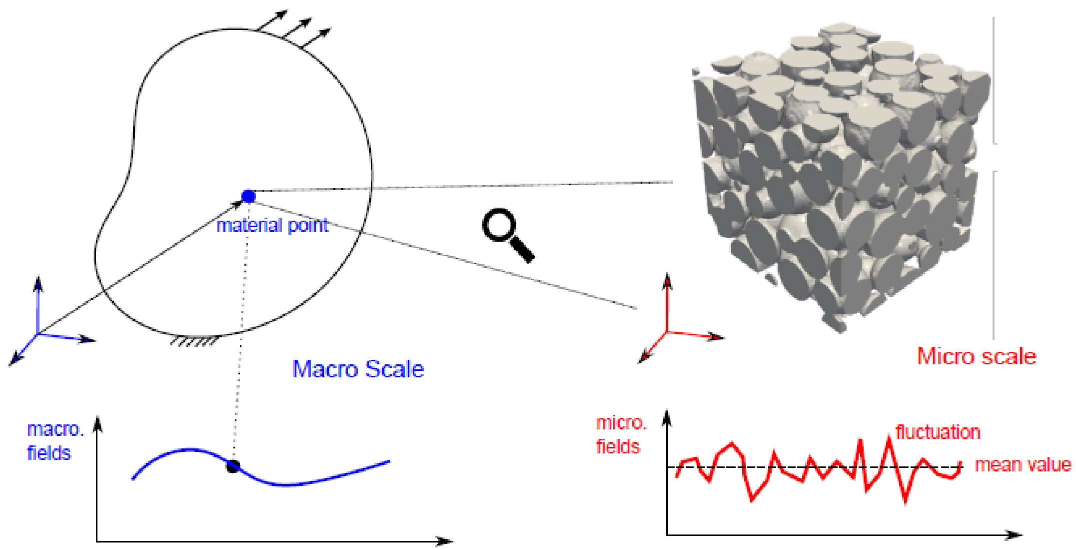
§ 4 Homogenization in Linear Elasticity

- Real Material: inhomogeneous, defects or nature or designed microstructures exist on different length scales.
- Ideal Material: simply homogenized on a certain scale
- Example: classic continuum mechanics $\rightarrow$ material point
- representative material properties $\approx$ homogenized properties of a RVE.

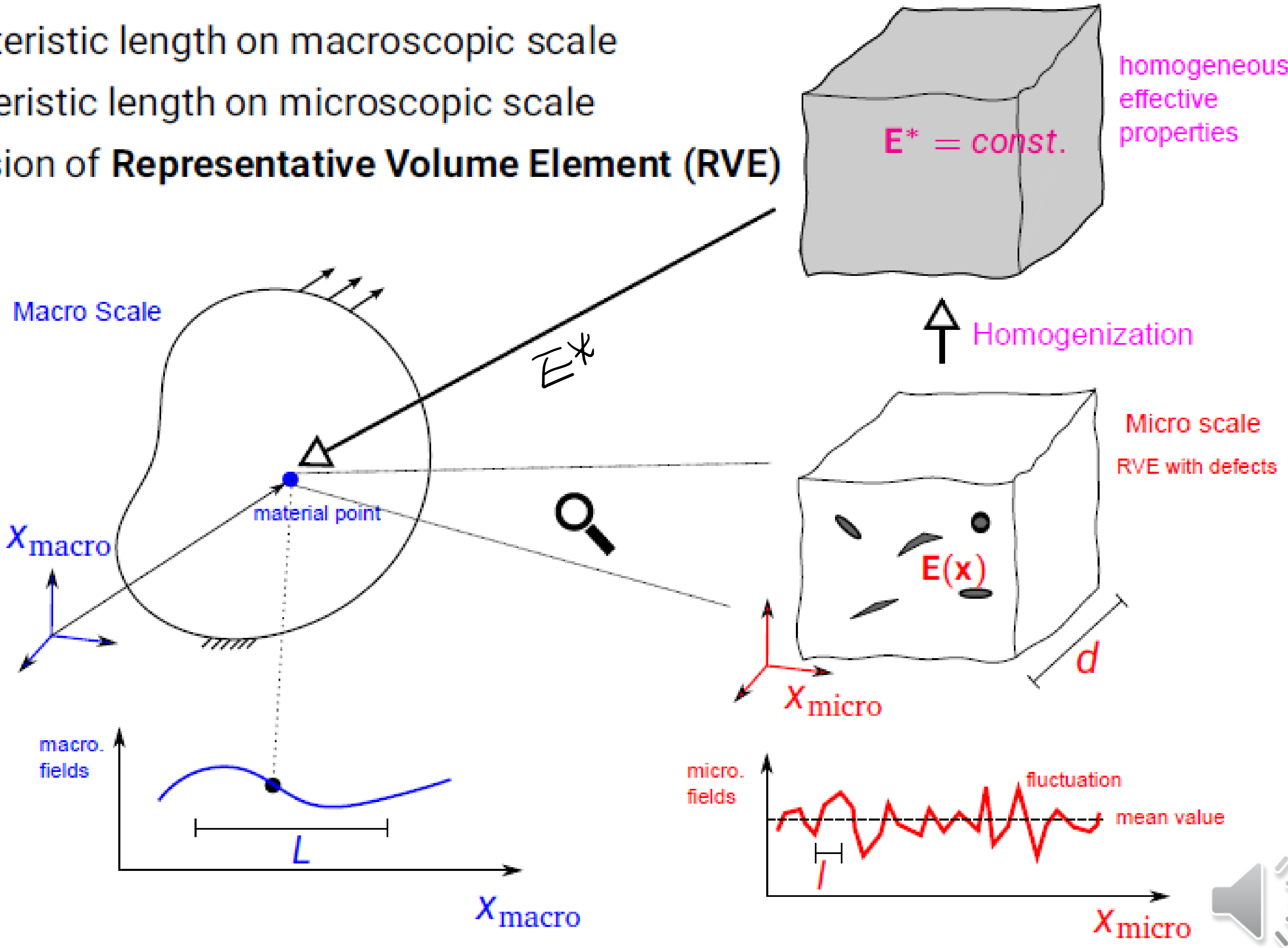


<u>characteristic scales:</u>

L: characteristic length on macroscopic scale

l: characteristic length on microscopic scale

d: dimension of **Representative Volume Element (RVE)**



L on macroscopic scale: characteristic length of an elastic body, at which the macroscopic quantities change.

l on microscopic scale: characteristic length of an elastic body, at which the microscopic fields change.

d: Dimension of the RVE, which contains a statistically representative distribution of defects.

Requirements of homogenization:

$$\boxed{l << d << L}$$

- Decoupling of scales
- Fluctuation of fields on microscopic scale is not appreciated on the macroscopic scale, but only the mean values.
- The gradient (L) of the macroscopic fields is not noticeable on the microscopic scale.
- Volume $V \sim d^3$ (RVE) = statistic representative for microstructure $\rightarrow$ includes characteristic defects ($d >> l$)

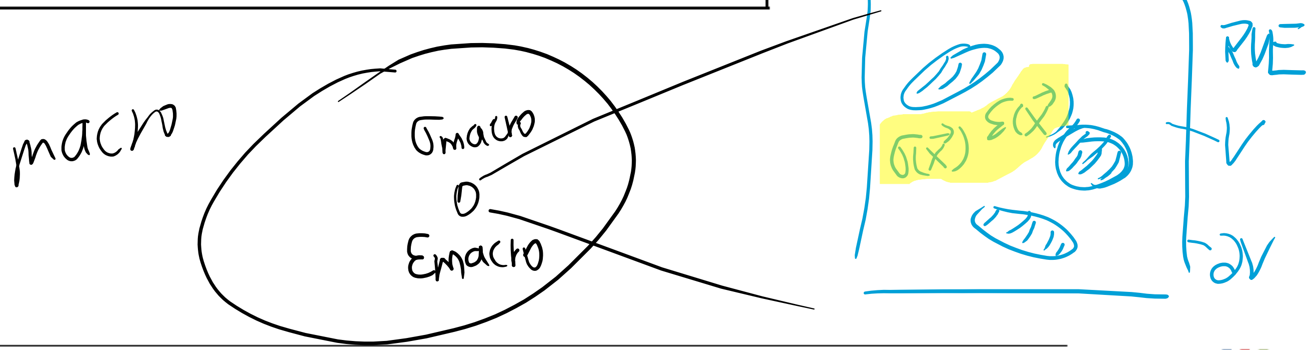
- It holds also for experimental determination of macroscopic material properties: Probes must be large enough, so that macroscopic material properties experimentally can be measured.
- In the case of periodic microstructure: *d* is given through the period.
- Restrictions:
  - at crack tips with singularities ($L \to 0$)
  - Strain localization
  - Nanomechanics
  - Gradient materials — $\sigma_{ij} \sim \varepsilon_{ij}, \nabla \varepsilon_{ij}$

§ 4.1 Mean values

Denote the stress and the strain fields in RVE V on the microscopic scale by $\boldsymbol{\sigma}(\mathbf{x}), \varepsilon(\mathbf{x})$, where $\mathbf{x}$ is the local coordinate of a point in RVE. The volumetric mean values of these fields are the stress and the strain quantity for the point on the macroscopic scale, to which the RVE is associated to. In other words,

$$\boldsymbol{\sigma}_{\text{macro}} = \frac{1}{V}\int_V \boldsymbol{\sigma}(\mathbf{x})\mathrm{d}V =< \boldsymbol{\sigma} >$$

$$\varepsilon_{\text{macro}} = \frac{1}{V}\int_V \varepsilon(\mathbf{x})\mathrm{d}V =< \varepsilon >$$



One can formulate $<\boldsymbol{\sigma}>$ as following, when no volume force is present:

$$<\sigma_{ij}>=\frac{1}{V}\int_V \overset{\sigma_{ij}}{\sigma_{ik}\delta_{jk}}\,\mathrm{d}V=\frac{1}{V}\int_V \sigma_{ik}x_{j,k}\,\mathrm{d}V=\frac{1}{V}\int_V\left[(\sigma_{ik}x_j)_{,k}-\overset{0}{\sigma_{ik,k}}x_j\right]\mathrm{d}V$$

$$=\frac{1}{V}\int_V(\sigma_{ik}x_j)_{,k}\,\mathrm{d}V=\frac{1}{V}\int_{\partial V}\sigma_{ik}n_k x_j\,\mathrm{d}A=\frac{1}{V}\int_{\partial V}t_i x_j\,\mathrm{d}A$$

Thereby stress equilibrium $\sigma_{ik,k}=0$ and the Gauß law is applied.

The mean values $<\varepsilon>$ can also be rewritten as follows:

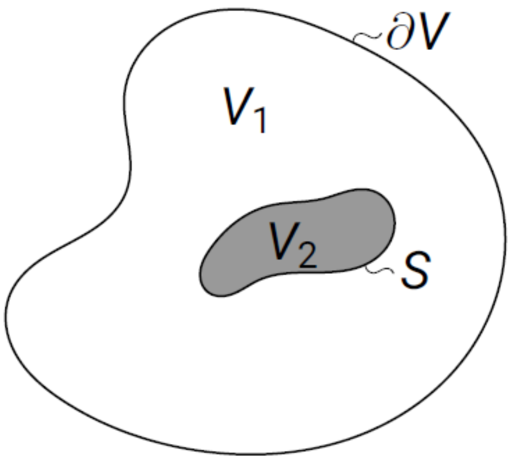
$$<\varepsilon_{ij}>=\frac{1}{2V}\int_V(u_{i,j}+u_{j,i})\mathrm{d}V=\frac{1}{2V}\int_{\partial V}(u_i n_j+u_j n_i)\mathrm{d}A$$

It means that the macroscopic fields $<\boldsymbol{\sigma}>,<\boldsymbol{\varepsilon}>$ can also be reformulated as integrals along the boundary of RVEs ∂V:

$$<\sigma_{ij}>=\frac{1}{V}\int_{\partial V}t_i x_j\,\mathrm{d}A \qquad <\boldsymbol{\sigma}>=\frac{1}{V}\int_{\partial V}\mathbf{t}\otimes\mathbf{x}\,\mathrm{d}A$$

$$<\varepsilon_{ij}>=\frac{1}{2V}\int_{\partial V}(u_i n_j+u_j n_i)\mathrm{d}A \qquad <\boldsymbol{\varepsilon}>=\frac{1}{2V}\int_{\partial V}(\mathbf{u}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{u})\mathrm{d}A \tag{4.1}$$

In fact, the Gauß law can only be applied when the fields can be differentiated. But it can be shown that the last two equations hold also for the cases of inhomogeneities with coherent interfaces.



$$V < \sigma_{ij} >= \int_V \sigma_{ij} \mathrm{d}V = \int_{V_1} \sigma_{ij} \mathrm{d}V + \int_{V_2} \sigma_{ij} \mathrm{d}V$$

$$= \int_{\partial V} t_i x_j \mathrm{d}A + \int_S (t_i^{(2)} - t_i^{(1)}) x_j \mathrm{d}A$$

$$2V < \varepsilon_{ij} >= \int_{V_1} + \int_{V_2} (u_i n_j + u_j n_i) \mathrm{d}V$$

$$= \int_{\partial V} (u_i n_j + u_j n_i) \mathrm{d}A +$$

$$+ \int_S \left[(u_i^{(2)} - u_i^{(1)}) n_j + (u_j^{(2)} - u_j^{(1)}) n_i \right] \mathrm{d}A$$

For coherent interfaces S it has $t_i^{(2)} = t_i^{(1)}$ and $u_i^{(2)} = u_i^{(1)}$. It follows that Eq. (4.1). It should be commented that the equations (4.1) hold also for cracks and holes.

Comments:

- Discrete phases: Volume parts V_α $(\alpha = 1, 2, ..., n)$ with $\mathbf{E}_\alpha$

$$<\boldsymbol{\sigma}>=\frac{1}{V}\int_V \boldsymbol{\sigma}\mathrm{d}V = \sum_{\alpha=1}^{n}\frac{1}{V}\int_{V_\alpha}\boldsymbol{\sigma}\mathrm{d}V = \sum_{\alpha=1}^{n}\frac{V_\alpha}{V}\frac{1}{V_\alpha}\int_{V_\alpha}\boldsymbol{\sigma}\mathrm{d}V = \sum_{\alpha=1}^{n} c_\alpha <\boldsymbol{\sigma}>_\alpha$$

where $c_\alpha = V_\alpha / V$ of the volume of the α phase, and $<\boldsymbol{\sigma}>_\alpha = \int_{V_\alpha}\boldsymbol{\sigma}\mathrm{d}V$ are the mean values of the α phase. It is commented that $\sum_\alpha c_\alpha = 1$.

Likewise,

$$<\varepsilon>=\sum_{\alpha=1}^{n} c_\alpha <\varepsilon>_\alpha$$

§ 4.2 Effective elastic constants

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Micro scale: $\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \qquad \sigma_{ij} = E_{ijkl}\varepsilon_{kl}$

Macro scale: $< \boldsymbol{\sigma} >= \mathbf{E}^* :< \boldsymbol{\varepsilon} > \qquad < \sigma_{ij} >= E^*_{ijkl} < \varepsilon_{kl} >$

$\boxed{\textit{Hill condition}}$: Mean values of strain energy densities must be equal !

$$< U >=< \frac{1}{2}\varepsilon_{ij}E_{ijkl}\varepsilon_{kl} >= \frac{1}{2} < \varepsilon_{ij} > E^*_{ijkl} < \varepsilon_{kl} >$$

Insertion by $E_{ijkl}\varepsilon_{kl} = \sigma_{ij}$ and $E^*_{ijkl} < \varepsilon_{kl} >=< \sigma_{ij} >$ leads to

$$\boxed{< \sigma_{ij}\varepsilon_{ij} > \; = \; < \sigma_{ij} >< \varepsilon_{ij} >}$$

energy denstiy on the micro scale = *energy density on the macro scale*

The difference between the mean values can be expressed by the fluctuation fields $\tilde{\sigma}_{ij}, \tilde{\varepsilon}_{ij}$:

$$\tilde{\sigma}_{ij} = \sigma_{ij} - < \sigma_{ij} >, \qquad \tilde{\varepsilon}_{ij} = \varepsilon_{ij} - < \varepsilon_{ij} >$$

From the first equation,

$$< \tilde{\sigma}_{ij} > = < \sigma_{ij} > - << \sigma_{ij} >> = < \sigma_{ij} > - < \sigma_{ij} > = 0$$

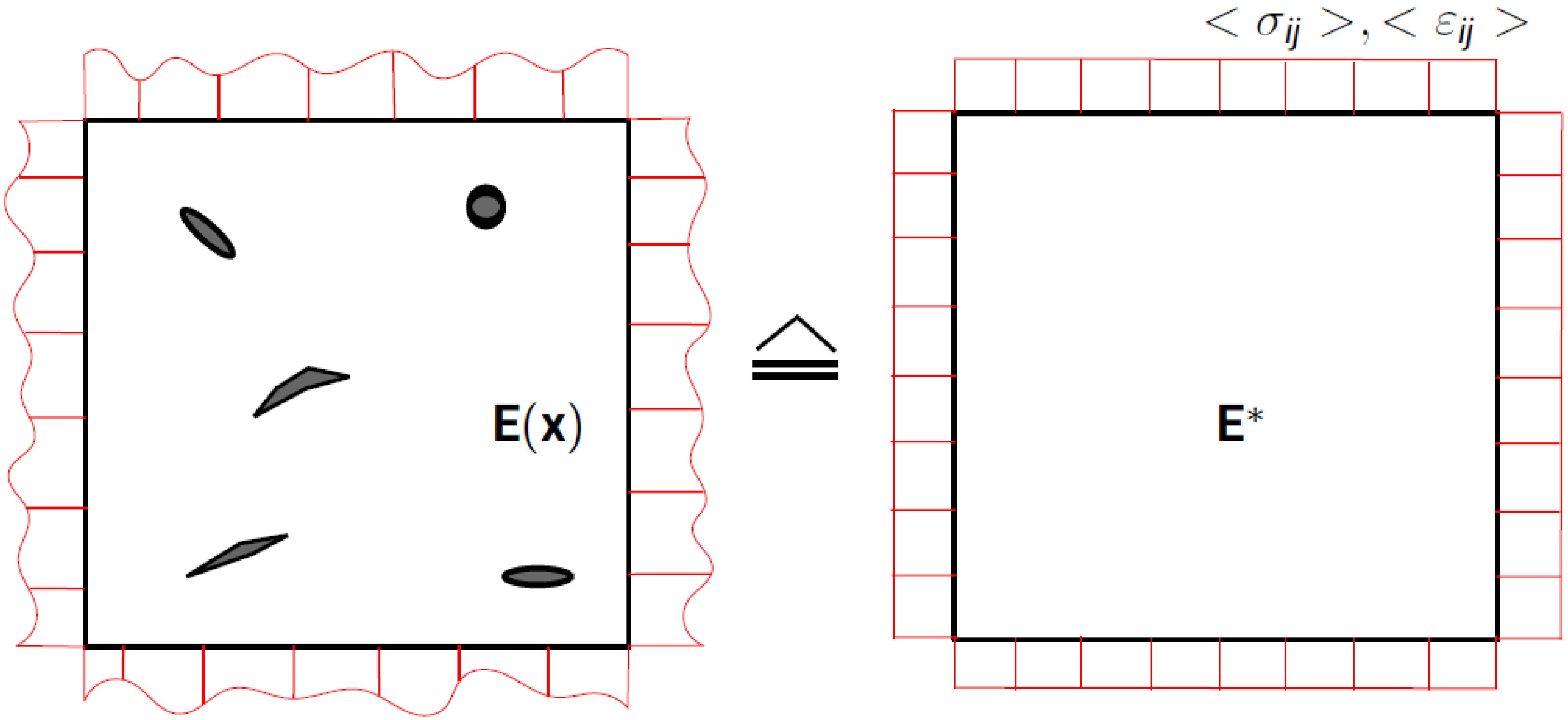
Similarly, one has $< \tilde{\varepsilon}_{ij} > = 0$. In use of the Hill-condition, one can prove that the fluctuation fields provide no work. In other words,

$$\boxed{< \tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\varepsilon}} > = < \tilde{\boldsymbol{\sigma}} > : < \tilde{\boldsymbol{\varepsilon}} > \qquad \text{or} \qquad < \tilde{\sigma}_{ij} \tilde{\varepsilon}_{ij} > = < \tilde{\sigma}_{ij} > < \tilde{\varepsilon}_{ij} >,}$$

and that

$$\boxed{< \tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\varepsilon}} > = < \tilde{\boldsymbol{\sigma}} > : < \tilde{\boldsymbol{\varepsilon}} > = \frac{1}{V} \int_{\partial V} \tilde{u}_i \tilde{t}_i \, \mathrm{d}A = 0.} \qquad (4.2)$$

in which $\tilde{u}_i = u_i - < \varepsilon_{ij} > x_j$, $\quad \tilde{t}_i = (\sigma_{ik} - < \sigma_{ik} >) n_k$.



Micro (inhomogeneous)

Macro (homogeneous)

$$< \tilde{\boldsymbol{\sigma}} : \tilde{\varepsilon} > = < \tilde{\boldsymbol{\sigma}} > : < \tilde{\varepsilon} > = \frac{1}{V} \int_{\partial V} \tilde{u}_i \tilde{t}_i \mathrm{d}A = 0.$$

in which $\tilde{u}_i = u_i - < \varepsilon_{ij} > x_j, \quad \tilde{t}_i = (\sigma_{ik} - < \sigma_{ik} >) n_k.$

In other words, $\tilde{u}_i = u_i - <\varepsilon_{ij}> x_j$ and $\tilde{t}_i = (\sigma_{ik} - <\sigma_{ik}>)n_k$ are two possible sufficient conditions, which ensure the Hill condition. $\rightarrow$ Two possible BCs for RVE

a) linear displacement on ∂V of the RVE

$$u_i = \varepsilon^0_{ij} x_j, \qquad \varepsilon^0_{ij} = const., \qquad \text{on } \partial V$$

One can calculate $<\varepsilon_{ij}>$ according to the definition of the average strain,

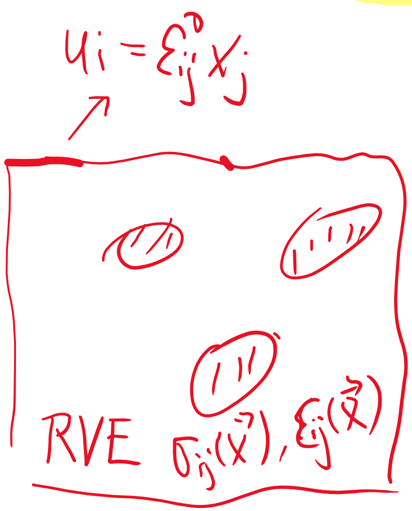
$$<\varepsilon_{ij}> = \frac{1}{V}\int_V \varepsilon_{ij} \mathrm{d}V = \underline{\frac{1}{2V}\int_{\partial V}(u_i n_j + u_j n_i)\mathrm{d}A} \Leftarrow \text{Eq.(4.1)}$$

$$= \frac{1}{2V}\varepsilon^0_{ik}\int_{\partial V} x_k n_j \mathrm{d}A + \frac{1}{2V}\varepsilon^0_{jk}\int_{\partial V} x_k n_i \mathrm{d}A$$

$$= \frac{1}{2V}\varepsilon^0_{ik}\int_V x_{k,j} \mathrm{d}V + \frac{1}{2V}\varepsilon^0_{jk}\int_V x_{k,i} \mathrm{d}V$$

$$= \frac{1}{2V}\varepsilon^0_{ik}\int_V \delta_{kj} \mathrm{d}V + \frac{1}{2V}\varepsilon^0_{jk}\int_V \delta_{ki} \mathrm{d}V$$

$$= \frac{1}{2}\varepsilon^0_{ik}\delta_{kj} + \frac{1}{2}\varepsilon^0_{jk}\delta_{ki} = \frac{1}{2}\varepsilon^0_{ij} + \frac{1}{2}\varepsilon^0_{ji} = \varepsilon^0_{ij}$$



The average strain theorem:

Assume $u_i(\mathbf{x})$ is continuous in RVE and $u_i = \varepsilon_{ij}^0 x_j$ on the boundary of RVE with an arbitrary constant tensor ε_{ij}^0. Then

$$\boxed{< \varepsilon_{ij} >= \varepsilon_{ij}^0, \qquad \text{or} \qquad < \boldsymbol{\varepsilon} >= \boldsymbol{\varepsilon}^0}$$

- Strain on the macro scale is known in this case.
- Replacement of $u_i = \varepsilon_{ij}^0 x_j$ into ∂V in Eq. (4.2) shows that using this linear displacement boundary condition ensures that the Hill-condition is automatically fulfilled.
- For numerical determination of $\mathbf{E}^*$ this boundary condition is suitable. Given the components of ε_{ij}^0, the stresses at boundary can be determined and thus the mean value of the stresses.

b) Uniform traction on ∂V

$$t_i = \sigma_{ij}^0 n_j, \qquad \sigma_{ij}^0 = const. \quad \text{on } \partial V$$

The mean stress $< \sigma_{ij} >$ can be calculated through Eq. $(4.1)_1$,

$$< \sigma_{ij} > = \frac{1}{V}\int_V \sigma_{ij} \mathrm{d}V = \frac{1}{V}\int_{\partial V} t_i x_j \mathrm{d}A$$

$$= \frac{1}{V}\sigma_{ik}^0 \int_{\partial V} n_k x_j \mathrm{d}A = \frac{1}{V}\sigma_{ik}^0 \int_V x_{j,k} \mathrm{d}A$$

$$= \frac{1}{V}\sigma_{ik}^0 \int_V \delta_{jk} \mathrm{d}A = \sigma_{ik}^0 \delta_{jk} = \sigma_{ij}^0$$

$t_i = \sigma_{ij}^0 n_j$

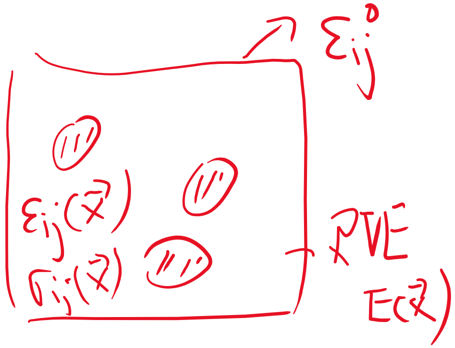
The average stress theorem:

Assume $\sigma_{ij,j} = 0$ in V (no volume force) and $t_i = \sigma_{ij}^0 n_j$ on ∂V with arbitrary $\sigma_{ij}^0 = const.$. Then

$$\boxed{<\sigma_{ij}> = \sigma_{ij}^0, \qquad \text{or} \qquad <\boldsymbol{\sigma}> = \boldsymbol{\sigma}^0}$$

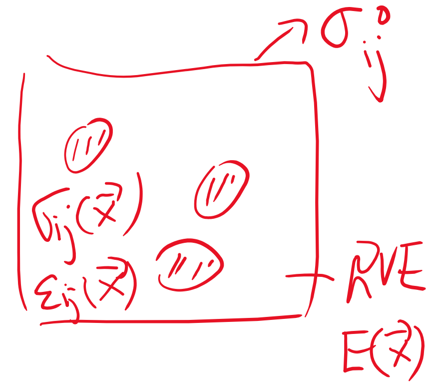
- Stress on the macroscopic scale is known in this case.
- Hill-condition is fulfilled automatically.
- It is suitable for numerical homogenization: Given σ_{ij}^0, the strain $\varepsilon_{ij}(\vec{x})$ or the displacement at boundary can be determined, and thus $<\varepsilon_{ij}>$. From $<\sigma_{ij}> = E_{ijkl}^* <\varepsilon_{ij}>$ the effective stiffness tensor E_{ijkl}^* can be determined then.

Intentionally one introduces the so-called Influence tensor, which is the normalized quantities of the solutions of boundary value problems a) or b):



a) $$\varepsilon(\vec{x}) = \mathbf{A}(\vec{x}) : \varepsilon^0$$

b) $$\sigma(\vec{x}) = \mathbf{B}(\vec{x}) : \sigma^0$$



Insertion in definition of $\mathbf{E}^*$ leads to:

a) $$< \sigma > = < \mathbf{E} : \varepsilon > = < \mathbf{E} : \mathbf{A} : \varepsilon^0 > = < \mathbf{E} : \mathbf{A} > : \varepsilon^0$$

$$\rightarrow \quad \boxed{\mathbf{E}^* = < \mathbf{E} : \mathbf{A} >} \qquad = \langle E : A \rangle : \langle \varepsilon \rangle \qquad (4.3)$$

b) $$< \varepsilon > = < \mathbf{E}^{-1} : \sigma > = < \mathbf{E}^{-1} : \mathbf{B} : \sigma^0 > = < \mathbf{E}^{-1} : \mathbf{B} > : \sigma^0 = (\mathbf{E}^*)^{-1} : \sigma^0$$

$\langle \sigma \rangle$

$$\rightarrow \quad \boxed{\mathbf{E}^* = < \mathbf{E}^{-1} : \mathbf{B} >^{-1}} \qquad (4.4)$$

Comments:

$E^* = \langle E : A \rangle$ $\quad$ $E^* = \langle E^{-1} : B \rangle^{-1}$

- The macroscopic property $\mathbf{E}^*$ or $(\mathbf{E}^*)^{-1}$ are weighted mean values!
  Attention: $\mathbf{E}^* \neq < \mathbf{E} >$, $(\mathbf{E}^*)^{-1} \neq < \mathbf{E}^{-1} >$
- For a RVE both of the boundary value problems should lead to the same results of effective stiffness tensor. The boundary conditions should play no role.
- It serves as Test for the quality of RVE.
- $< \varepsilon > = < \mathbf{A} : \varepsilon^0 > = < \mathbf{A} > : \varepsilon^0$. On the other hand, one has from the average strain theorem $< \varepsilon > = \varepsilon^0$. It follows that

$\varepsilon(\vec{x}) = A(\vec{x}) : \varepsilon^0$

$$< \mathbf{A} > = \mathbf{I}$$

$\langle \varepsilon \rangle = \langle A : \varepsilon^0 \rangle = \underbrace{\langle A \rangle}_{I} : \underbrace{\varepsilon^0}_{\langle \varepsilon \rangle}$

- Likewise, the following equation holds

$\sigma(\vec{x}) = B(\vec{x}) : \sigma^0$

$$< \mathbf{B} > = \mathbf{I}$$

$\langle \sigma \rangle = \langle B \rangle : \sigma^0 = \underbrace{\langle B \rangle}_{I} : \langle \sigma \rangle$

- Approximations of $\mathbf{A}$ and $\mathbf{B}$ lead to approximation for $\mathbf{E}^*$.

§ 4.3 Multi-phase composites

Without general restrictions we consider a 2-Phase material: Matrix M with $\mathbf{E}_M = const.$ and inhomogeneity I with $\mathbf{E}_I = const.$. It follows that

$$\mathbf{E}^* = <\mathbf{E} : \mathbf{A}> = \frac{1}{V}\int_V \mathbf{E}(\mathbf{x}) : \mathbf{A}(\mathbf{x}) \mathrm{d}V$$

$$= \mathbf{E}_M \frac{V_M}{V}\frac{1}{V_M}\int_{V_M} \mathbf{A} \mathrm{d}V + \mathbf{E}_I \frac{V_I}{V}\frac{1}{V_I}\int_{V_I} \mathbf{A} \mathrm{d}V$$

$$= \mathbf{E}_M c_M \mathbf{A}_M + \mathbf{E}_I c_I \mathbf{A}_I$$

where $c_M = \dfrac{V_M}{V}$, $c_I = \dfrac{V_I}{V}$ are the volume concentration of the matrix and the inhomogeneity, respectively.

$< \circ > = c_M < \circ >_M + c_I < \circ >_I$

$< A > = c_M < A >_M + c_I < A >_I$

Introduce $\mathbf{A}_M = <\mathbf{A}>_M = \frac{1}{V_M}\int_{V_M}\mathbf{A}\,dV$, $\mathbf{A}_I = <\mathbf{A}>_I = \frac{1}{V_I}\int_{V_I}\mathbf{A}\,dV$.

It can be obtained from $<\mathbf{A}> = c_M\mathbf{A}_M + c_I\mathbf{A}_I = \mathbf{I}$. $\rightsquigarrow$ $c_M A_M = I - c_I A_I$

$E^* = E_M c_M A_M + E_I c_I A_I$

Thus

$$\boxed{\mathbf{E}^* = \mathbf{E}_M(\mathbf{I} - c_I\mathbf{A}_I) + \mathbf{E}_I c_I\mathbf{A}_I = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I}$$

- It should be noted that: $<\varepsilon>_I = <\mathbf{A} : \varepsilon^0>_I = <\mathbf{A}>_I : \varepsilon^0 = \mathbf{A}_I : \varepsilon^0$
- Only the influence tensor for one phase is required.
- For a n-phase material one requires influence tensors of $n-1$ phases.
- Similarly for the boundary type b):

$$\boxed{\mathbf{E}^* = \left[(\mathbf{E}_M)^{-1} + c_I(\mathbf{E}_I^{-1} - \mathbf{E}_M^{-1}) : \mathbf{B}_I\right]^{-1}}$$

$$E^* = E_M + \underbrace{c_I^{(1)}(E_I^{(1)} - E_M)A_I^{(1)} + \dots + c_I^{(n-1)}(E_I^{(n-1)} - E_M)A_I^{(n-1)}}_{n-1 \text{ phases}}$$

§ 4.4 Voigt/Reuss Approximationen

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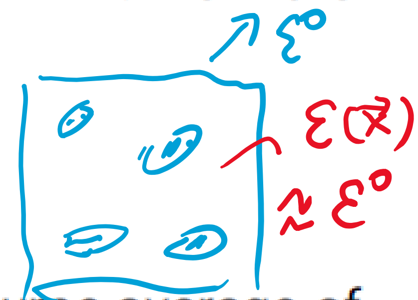
Constant strain approximation (Voigt, 1889):

Assume the strain field in boundary value problem a) is homogeneous. In other words, from $\varepsilon(\mathbf{x}) = const. = \varepsilon^0 =< \varepsilon >$ in V, one has $\mathbf{A} = \mathbf{I}$. From the general solution (4.3), one has



$$\boxed{\mathbf{E}^*_{Voigt} =< \mathbf{E} : \mathbf{A} >=< \mathbf{E} >}$$

It means that the effective stiffness tensor is simply the volume average of inhomogeneous stiffness in the system, according to the Voigt-Approximation. For a 2-Phase Material, it holds

$$\mathbf{E}^*_{Voigt} = c_M \mathbf{E}_M + c_I \mathbf{E}_I$$

where c_M, c_I is the volume concentration of the matrix and inhomogeneity, respectively, and $c_M = 1 - c_I$.

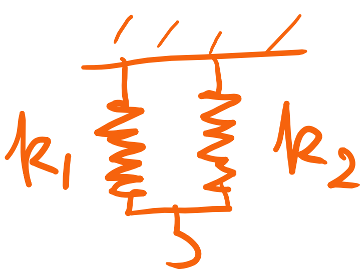
When both phases are isotropic, the effective stiffness tensor will hereby also be isotropic

$E^*_{voigt} = c_M E_M + c_I E_I$

$$\boxed{K^*_{Voigt} = c_M K_M + c_I K_I, \qquad \mu^*_{Voigt} = c_M \mu_M + c_I \mu_I}$$

where K_M, K_I are the bulk modulus of the matrix and of the inhomogeneity, respectively. And μ_M, μ_I are their shear modulus.
It should be noted that the effective elastic constants E^*_{Voigt} and ν^*_{Voigt} are not the mean value of the corresponding quantities. They should be calculated from K^*_{Voigt} and μ^*_{Voigt} through the following equations

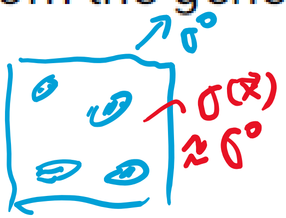
$$E^* = \frac{9K^*\mu^*}{3K^* + \mu^*}, \qquad \nu^* = \frac{3K^* - 2\mu^*}{2(3K^* + \mu^*)}$$



$k = k_1 + k_2$

Constant stress approximation (Reuss, 1929):

It is assumed that the stress field in the boundary value problem b) is homogeneous, i.e. $\boldsymbol{\sigma}(\mathbf{x}) = const. = \boldsymbol{\sigma}^0 =< \boldsymbol{\sigma} >$ in V. It follows that $\mathbf{B} = \mathbf{I}$. From the general expression (4.4) one has



$$\boxed{\mathbf{E}^*_{Reuss} =< \mathbf{E}^{-1} : \mathbf{B} >^{-1} =< \mathbf{E}^{-1} >^{-1}}$$

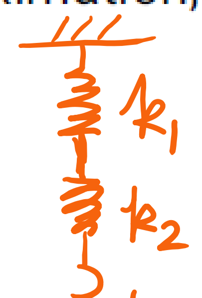
$E^{*\,-1}_{Reuss} = \langle E^{-1} \rangle$

Thus, in this approach, the effective compliance tensor of the heterogeneous material is the volume average of the compliance tensors of the phases in the material. For a 2-phase Material,

$$(\mathbf{E}^*_{Reuss})^{-1} = c_M \mathbf{E}_M^{-1} + c_I \mathbf{E}_I^{-1}$$

If the two phases are isotropic, according to the Reuss-Approximation, the effective elastic properties are also isotropic:

$$\boxed{\frac{1}{K^*_{Reuss}} = \frac{c_M}{K_M} + \frac{c_I}{K_I}, \qquad \frac{1}{\mu^*_{Reuss}} = \frac{c_M}{\mu_M} + \frac{c_I}{\mu_I}}$$



$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

Comments: The Voigt and the Reuss Approximation presents the upper and the lower bound of the effective estimations.

Proof: **E** and $\mathbf{E}^{-1}$ are positive defined. In other words, it holds for any arbitrary admissible strain $\varepsilon - <\varepsilon>$:

$-\varepsilon : E : <\varepsilon> - <\varepsilon> : E : \varepsilon$

$$(\varepsilon - <\varepsilon>) : \mathbf{E} : (\varepsilon - <\varepsilon>) \geq 0$$

$$\varepsilon : \mathbf{E} : \varepsilon + <\varepsilon> : \mathbf{E} : <\varepsilon> -2 <\varepsilon> : \mathbf{E} : \varepsilon \geq 0$$

Taking the mean values of the previous equations,

$$<\varepsilon : \mathbf{E} : \varepsilon> + <\varepsilon> : <\mathbf{E}> : <\varepsilon> -2 <\varepsilon> : <\mathbf{E} : \varepsilon> \geq 0$$

$$<\varepsilon> : \mathbf{E}^* : <\varepsilon> + <\varepsilon> : <\mathbf{E}> : <\varepsilon> -2 <\varepsilon> : <\sigma> \geq 0$$

$$<\varepsilon> : \mathbf{E}^* : <\varepsilon> + <\varepsilon> : <\mathbf{E}> : <\varepsilon> -2 <\varepsilon> : \mathbf{E}^* : <\varepsilon> \geq 0$$

$<\varepsilon> : <E> : <\varepsilon> - <\varepsilon> : E^* : <\varepsilon> \geq 0$

where the Hill-condition $<\varepsilon : \mathbf{E} : \varepsilon> = <\varepsilon> : \mathbf{E}^* : <\varepsilon>$ and the definition $<\mathbf{E} : \varepsilon> = <\boldsymbol{\sigma}> = \mathbf{E}^* : <\varepsilon>$ was applied. Hence,

$$<\varepsilon> : (<\mathbf{E}> - \mathbf{E}^*) : <\varepsilon> \geq 0 \quad \rightarrow \quad <\mathbf{E}> = \boxed{\mathbf{E}^*_{Voigt} \geq \mathbf{E}^*}$$

$<E> \geq E^*$

$K^*_{voigt} > K^*$

$\mu^*_{voigt} > \mu^*$

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In other words, the Voigt Approximation leads to the upper bound.

Likewise,

$$(\boldsymbol{\sigma} - < \boldsymbol{\sigma} >) : \mathbf{E}^{-1} : (\boldsymbol{\sigma} - < \boldsymbol{\sigma} >) \geq 0$$

and similar derivation leads to

$$< \boldsymbol{\sigma} >: \left[< \mathbf{E}^{-1} > - (\mathbf{E}^*)^{-1} \right] :< \boldsymbol{\sigma} > \geq 0$$

or,

$$< \mathbf{E} >^{-1} \geq (\mathbf{E}^*)^{-1}, \qquad \text{d.h.} \qquad < \mathbf{E}^{-1} >^{-1} = \boxed{\mathbf{E}^*_{Reuss} \leq \mathbf{E}^*}$$

The Reuss Approximation represents the lower bound. In summary,

$$\boxed{\mathbf{E}^*_{Reuss} \leq \mathbf{E}^* \leq \mathbf{E}^*_{Voigt}}$$

$K^*_{Reuss} \leq K^* \leq K^*_{voigt}$

Pragmatically, one uses the mean of the Voigt and the Reuss approximation for the estimation

$$\mathbf{E}^* \approx \frac{1}{2}(\mathbf{E}^*_{Voigt} + \mathbf{E}^*_{Reuss})$$

§ 4.5 Dilute distribution approximation (DD)

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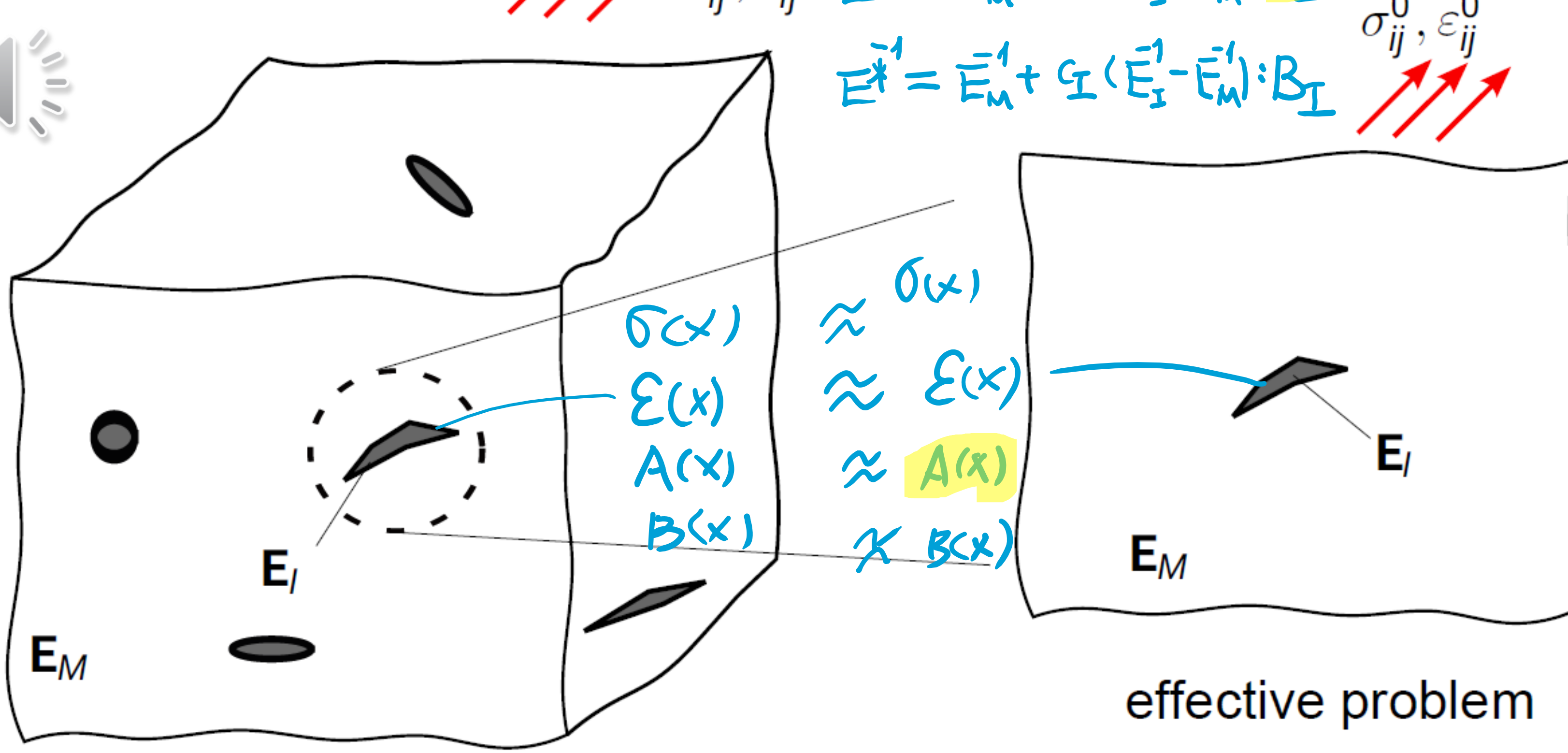
Assume:

- small volume concentration and distributed $\rightarrow$ defect interaction is ignored
- every defect feels only the external loading

$$\mathbb{E}^* = \mathbb{E}_M + c_I (\mathbb{E}_I - \mathbb{E}_M) : \mathbb{A}_I$$

$$\mathbb{E}^{*-1} = \mathbb{E}_M^{-1} + c_I (\mathbb{E}_I^{-1} - \mathbb{E}_M^{-1}) : \mathbb{B}_I$$



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$$E^* = E_M + c_I (E_I - E_M) : A_I$$

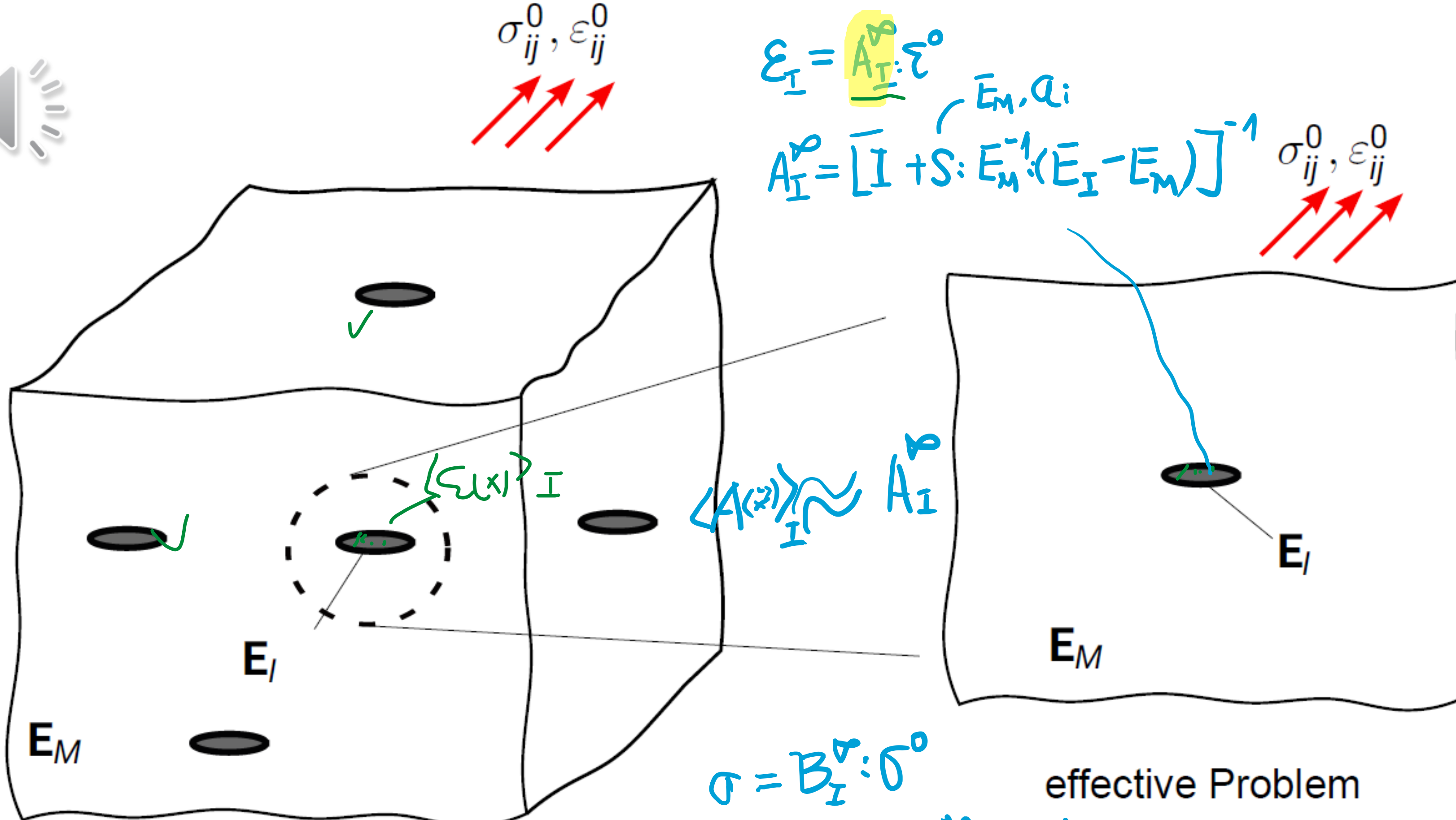
$$E^{*-1} = E_M^{-1} + c_I (E_I^{-1} - E_M^{-1}) : B_I$$

Ellipsoidal inhomogeneities (2 Phases, $c_I << 1$)

$$\varepsilon_I = A_I^\infty : \varepsilon^0$$

E_M, a_i

$$A_I^\infty = \left[I + S : E_M^{-1} : (E_I - E_M) \right]^{-1}$$



$$\langle A(x) \rangle_I \approx A_I^\infty$$

$$\sigma = B_I^\infty : \sigma^0$$

$$B \approx B_I^\infty = E_I : A_I^\infty : E_M^{-1}$$

In the subsection 4.3 we have obtained the general expression for $\mathbf{E}^*$ in linear displacement boundary a)

$$\mathbf{E}^* = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I$$

where $\mathbf{A}_I$ is the mean value of the influence tensor $\mathbf{A}$ in the inhomogeneities in the effective problem. The effective problem according to the DD method is exactly the problem, which we have considered in subsction 3.6. The influence tensor $\mathbf{A}_I^\infty$ is given in Eq. (3.3). It is known that $\mathbf{A}_I^\infty$ is uniform in the inhomogeneity, i.e.

$$\mathbf{A}_I = \mathbf{A}_I^\infty = \left[\mathbf{I} + \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M)\right]^{-1}$$

where $\mathbf{S}$ is the Eshelby Tensor of 4th. Note that $\mathbf{S}$ depends on $\mathbf{E}_M$ and the geometry of the ellipsoidal inhomogeneities, because the matrix material in the effective problem has $\mathbf{E}_M$. Insertion of $\mathbf{A}_I$ into the general expressions leads to the effective stiffness tensor

E_M, a_i

$$\boxed{\mathbf{E}_{DD}^{*(a)} = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \left[\mathbf{I} + \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M)\right]^{-1}} \qquad (4.5)$$

$$\mathbf{E}_{DD}^{*(a)} = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \left[\mathbf{I} + \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M)\right]^{-1}$$

- For $c_I = 0$, $\mathbf{E}_{DD}^{*(a)} = \mathbf{E}_M$. For $c_I = 1$, $\mathbf{E}_{DD}^{*(a)} \neq \mathbf{E}_I$. It should be commented that the DD approach is valid for $c_I << 1$.
- Linear dependency on c_I.
- Note: even if both phases are isotropic materials, the effective $\mathbf{E}^*$ can be anisotropic, due to the anisotropy of the geometry of the inhomogeneities.
- Similarly, one has for the boundary value problem b)

$$\mathbf{E}_{DD}^{*(b)} = \left[\mathbf{E}_M^{-1} + c_I(\mathbf{E}_I^{-1} - \mathbf{E}_M^{-1}) : \mathbf{B}_I\right]^{-1}$$

where

$$\mathbf{B}_I = \mathbf{B}_I^{\infty} = \mathbf{E}_I : \mathbf{A}_I^{\infty} : \mathbf{E}_M^{-1} = \mathbf{E}_I : \left[\mathbf{I} + \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M)\right]^{-1} : \mathbf{E}_M^{-1}$$

- $\mathbf{E}_{DD}^{*(a)} \neq \mathbf{E}_{DD}^{*(b)}$

- Isotropic case: $\mathbf{E}_I$, $\mathbf{E}_M$ are isotropic, and the inhomogeneities has spherical shape. For spherical inhomogeneities,

$$S_{ijkl} = \alpha\frac{1}{3}\delta_{ij}\delta_{kl} + \beta(I_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}), \quad \alpha = \frac{3K_M}{3K_M + 4\mu_M}, \quad \beta = \frac{6(K_M + 2\mu_M)}{5(3K_M + 4\mu_M)}$$

Insertion into Eq. (4.5) leads to

$$K_{DD}^{*(a)} = K_M + c_I\frac{(K_I - K_M)K_M}{K_M + \alpha(K_I - K_M)}, \qquad \mu_{DD}^{*(a)} = \mu_M + c_I\frac{(\mu_I - \mu_M)\mu_M}{\mu_M + \beta(\mu_I - \mu_M)}$$

It can be seen that K_{DD}^*, μ_{DD}^* depend linearly on c_I.

$$E_{DD}^{*(a)} = E_M + c_I(E_I - E_M) : [I + S : E_M^{-1} : (E_I - E_M)]^{-1}$$

$$E_M \mathrel{\hat{=}} (3K_M, 2\mu_M) \qquad I \mathrel{\hat{=}} (1, 1)$$

$$E_I \mathrel{\hat{=}} (3K_I, 2\mu_I) \qquad S \mathrel{\hat{=}} (\alpha, \beta)$$

$$K_{DD}^* = K_M + c_I(K_I - K_M)\frac{1}{1 + \alpha\frac{1}{K_M}(K_I - K_M)}$$

$$= K_M + c_I\frac{(K_I - K_M)K_M}{K_M + \alpha(K_I - K_M)}$$

In the case of hard particles ($K_I \to \infty, \mu_I \to \infty$) and incompressible matrix ($K_M \to \infty, \mu_M$), it holds that $\alpha \to 1$ and $\beta \to \dfrac{2}{5}$.

From the last two equations, one has

$$\mu_{DD}^{*(a)} \to (1 + \frac{5}{2}c_I)\mu_M, \qquad K_{DD}^{*(a)} \to \infty$$

$$K_{DD}^{*(a)} = K_M + c_I \frac{(K_I - K_M)K_M}{K_M + \alpha(K_I - K_M)} \to \infty$$

$$\alpha = \frac{3K_M}{3K_M + 4\mu_M} = \frac{3}{3 + 4\cancel{\frac{\mu_M}{K_M}}_{\to 0}} = 1$$

$$\beta = \frac{6(K_M + 2\mu_M)}{5(3K_M + 4\mu_M)} = \frac{6(1 + 2\cancel{\frac{\mu_M}{K_M}})}{5(3 + 4\cancel{\frac{\mu_M}{K_M}})} = \frac{2}{5}$$

$$\mu_{DD}^{*(a)} = \mu_M + c_I \frac{(\mu_I - \mu_M)\mu_M}{\mu_M + \beta(\mu_I - \mu_M)}$$

$$= \mu_M + c_I \frac{(1 - \cancel{\frac{\mu_M}{\mu_I}})\mu_M}{\cancel{\frac{\mu_M}{\mu_I}} + \beta(1 - \cancel{\frac{\mu_M}{\mu_I}})}$$

$$= \mu_M + c_I \frac{1}{\beta}\mu_M \qquad \frac{2}{5}$$

$$= (1 + \frac{5}{2}c_I)\mu_M$$

§ 4.6 Mori-Tanaka Approach (MT)

$E^* = E_M + c_I (E_I - E_M) : A_I$

In general, $\varepsilon(\mathbf{x}), \sigma(\mathbf{x})$ have fluctuation in the matrix around the mean value:

Voigt : $A_I \approx I$

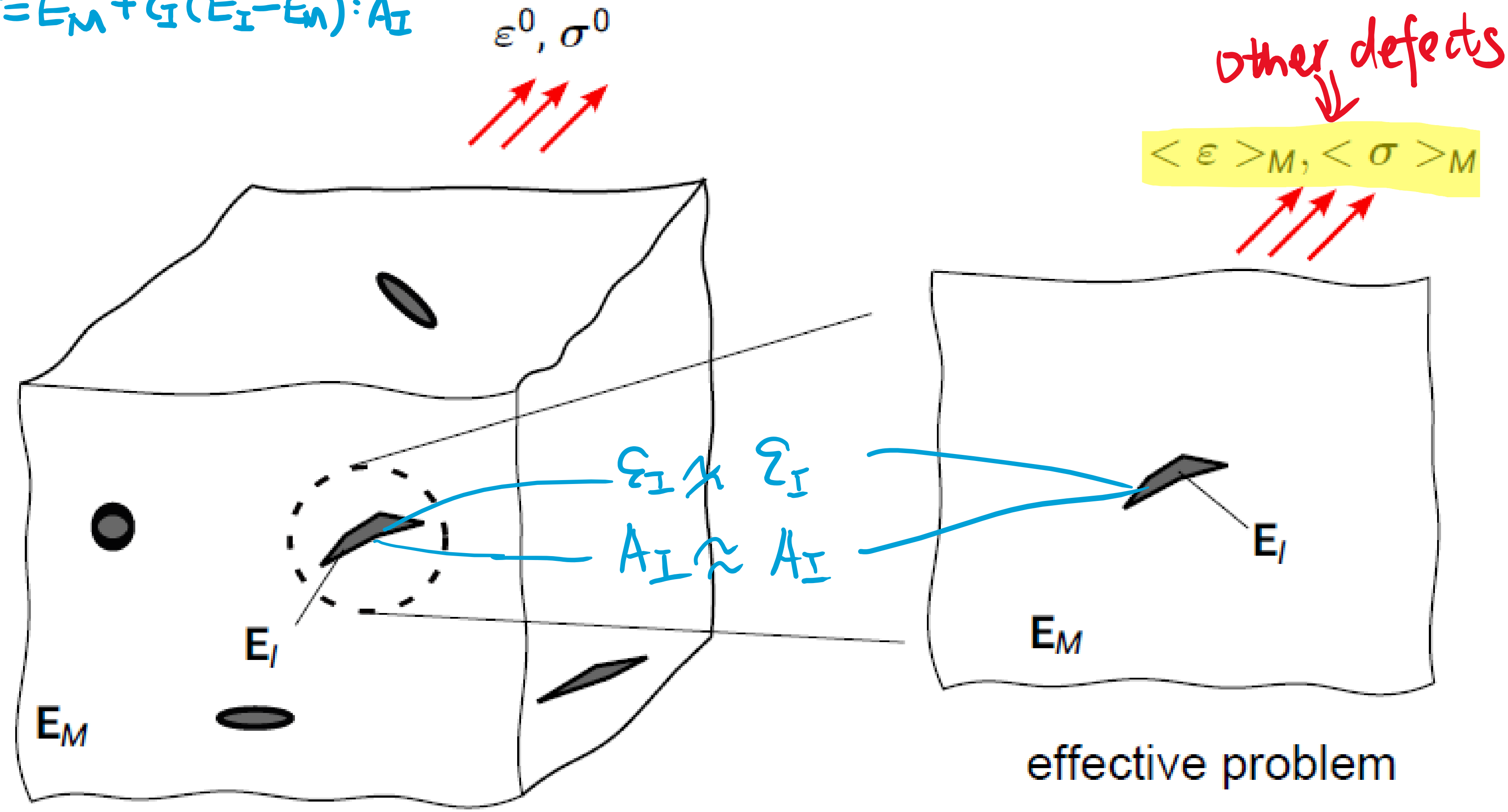
$$\varepsilon(\mathbf{x})|_M = <\varepsilon>_M + \tilde{\varepsilon}(\mathbf{x}), \qquad \sigma(\mathbf{x})|_M = <\sigma>_M + \tilde{\sigma}(\mathbf{x})$$

DD : $A_I \approx A_I^\infty$

T. Mori and K. Tanaka have developed in 1973 a new approach to evaluate the effective stiffness tensor. The basic idea is to consider the interaction between the defects/inhomogeneities through their influence on the mean values of the fields in the matrix ("mean field theory"). For the MT approach, the following assumptions or requirements should hold:

- Defects feel only the homogeneous field $<\varepsilon>_M$ and $<\sigma>_M$. They do not feel ε^0 or σ^0.
- $<\varepsilon>_M$ and $<\sigma>_M$ depend on other defects.
- Defects are isolated in the matrix.
- Fluctuations induced by defects decay in a sufficient distance.

$$E^* = E_M + G(E_I - E_M) : A_I$$



ε^0, σ^0
other defects
$< \varepsilon >_M, < \sigma >_M$
$\varepsilon_I \not\approx \varepsilon_I$
$A_I \approx A_I$
E_I
E_M
E_I
E_M
effective problem

In order to compare with the DD approach, the loading in the effective problem is not ε^0, $\boldsymbol{\sigma}^0$, but $<\varepsilon>_M$, $<\boldsymbol{\sigma}>_M$.

In the case of 2 phase composite with ellipsoidal inhomogeneities,

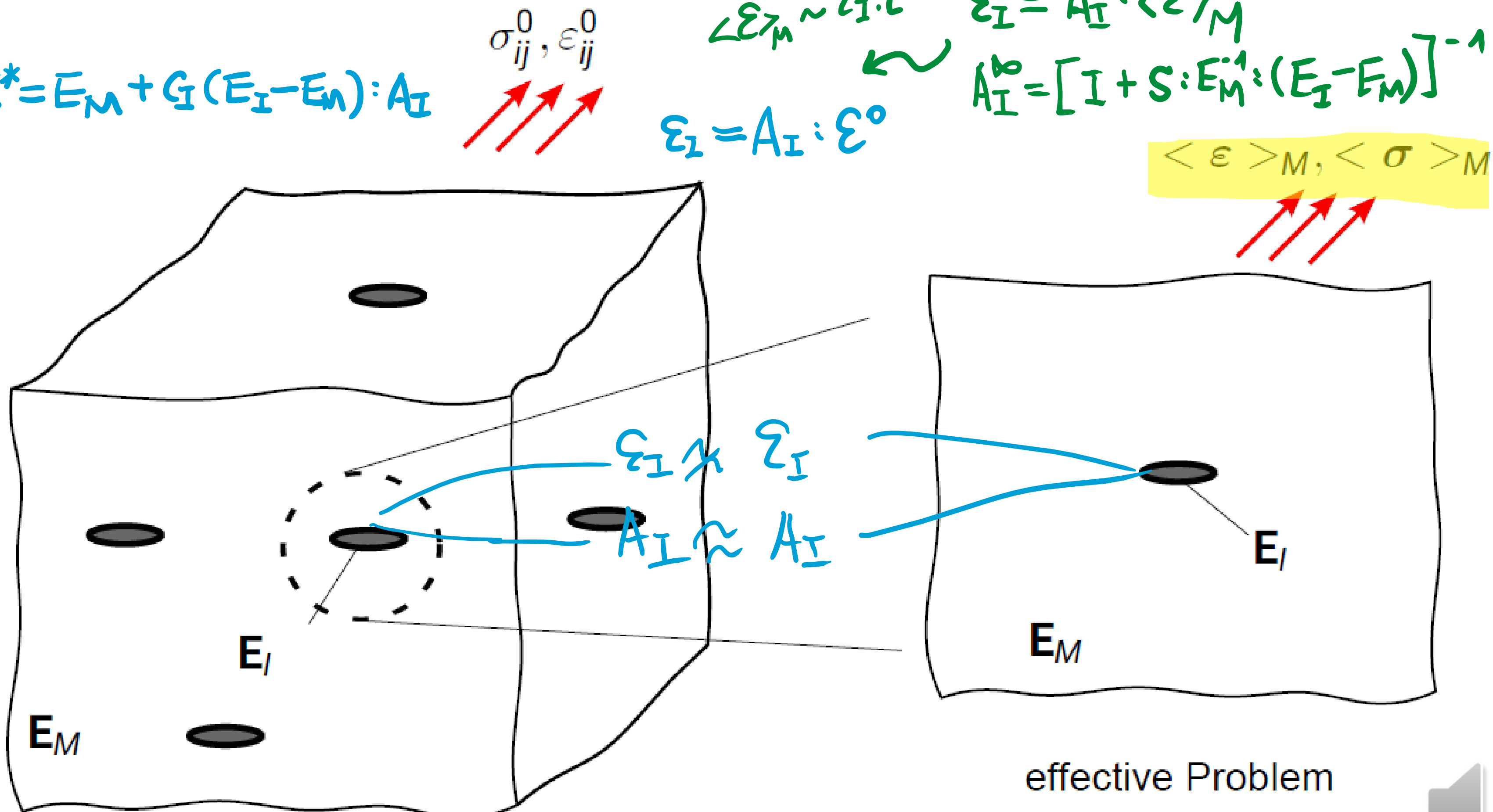
$E^* = E_M + c_I(E_I - E_M) : A_I$

$\langle\varepsilon\rangle_M \sim \varepsilon_I, \varepsilon^0$

$\varepsilon_I = A_I^\infty : \langle\varepsilon\rangle_M$

$A_I^\infty = [I + S : E_M^{-1} : (E_I - E_M)]^{-1}$

$\varepsilon_I = A_I : \varepsilon^0$



In the general expression $\mathbf{E}^* = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I$, $\mathbf{A}_I$ denotes the relation between strain in the inhomogeneities and the real external loading $\varepsilon^0, \boldsymbol{\sigma}^0$. For the boundary condition a) $\mathbf{A}_I$ is defined through $<\varepsilon>_I = \mathbf{A}_I : \varepsilon^0$. By the concept of equivalent eigenstrain, one obtains in the effective problem the relation between $<\varepsilon>_I$ and the effective field $<\varepsilon>_M$:

$$<\varepsilon>_I = \mathbf{A}_I^\infty : <\varepsilon>_M$$

where $\mathbf{A}_I^\infty$ is given in Eq. (3.3). Moreover, one obtains from

$$<\varepsilon> = c_I <\varepsilon>_I + c_M <\varepsilon>_M$$

$$<\varepsilon>_M = \frac{1}{c_M}(<\varepsilon> - c_I <\varepsilon>_I)$$

$$\rightarrow \quad <\varepsilon>_I = \frac{1}{c_M}\mathbf{A}_I^\infty : (<\varepsilon> - c_I <\varepsilon>_I)$$

$$\rightarrow \quad c_M(\mathbf{A}_I^\infty)^{-1} : <\varepsilon>_I + c_I <\varepsilon>_I = <\varepsilon>$$

$$\rightarrow \quad [c_M(\mathbf{A}_I^\infty)^{-1} + c_I\mathbf{I}] : <\varepsilon>_I = <\varepsilon>$$

$$\rightarrow \quad <\varepsilon>_I = [c_I\mathbf{I} + c_M(\mathbf{A}_I^\infty)^{-1}]^{-1} : <\varepsilon>$$

A_I^{MT}

ε^0

From the average strain theory, one has: $< \varepsilon >= \varepsilon^0$. It follows that

$$< \varepsilon >_I = \mathbf{A}_I^{MT} : \varepsilon^0$$

where

$$\mathbf{A}_I^{MT} = [c_I \mathbf{I} + c_M (\mathbf{A}_I^{\infty})^{-1}]^{-1}$$

One can replace $\mathbf{A}_I^{\infty}$ into the Eq. (3.3) and obtain

$$\mathbf{A}_I^{MT} = \left\{ c_I \mathbf{I} + c_M \left[\mathbf{I} + \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M) \right] \right\}^{-1}$$

$$\boxed{\mathbf{A}_I^{MT} = \left[\mathbf{I} + c_M \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M) \right]^{-1}}$$

$$A_I^{DD} = A_I^{\infty} = [I + S : E_M^{-1} : (E_I - E_M)]^{-1}$$

Note that the only difference between $\mathbf{A}_I^{MT}$ and $\mathbf{A}_I^{DD}$ in the DD approach exists only in the factor c_M in the second term. Thereby $\mathbf{S}$ depends on $\mathbf{E}_M$ and the geometry of the ellipsoidal inhomogeneities, because the matrix material in the effective problem has $\mathbf{E}_M$. Insertion into the general expressions leads to:

$$\boxed{\mathbf{E}_{MT}^{*(a)} = \mathbf{E}_M + c_I (\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I^{MT}}$$

$$\mathbf{E}_{MT}^{*(a)} = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I^{MT}$$

$$\mathbf{A}_I^{MT} = \left[\mathbf{I} + c_M \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M)\right]^{-1}$$

Comments:

- In constrast to the DD approach, the MT approach is valid for both extreme cases:

$$c_I \to 0: \qquad \mathbf{E}_{MT}^* = \mathbf{E}_M$$

$$c_I \to 1: \qquad \mathbf{E}_{MT}^* = \mathbf{E}_I$$

- Nonlinear dependency on c_I.
- $\mathbf{E}_{MT}^*$ is independent of the fact, whether $< \varepsilon >_M$ or $< \sigma >_M$ is used in the effective problem.

$$\mathbf{E}^{*(a)}_{MT} = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}^{MT}_I$$

$$\mathbf{A}^{MT}_I = \left[\mathbf{I} + c_M \mathbf{S} : \mathbf{E}_M^{-1} : (\mathbf{E}_I - \mathbf{E}_M)\right]^{-1}$$

$E_M \hat{=} (3K_M, 2\mu_M)$

$E_I \hat{=} (3K_I, 2\mu_I)$

$E^*_{MT} \hat{=} (3K^*_{MT}, 2\mu^*_{MT})$

$S \hat{=} (\alpha, \beta)$

- For the special case: isotropic $\mathbf{E}_I$, $\mathbf{E}_M$ and spherical inhomogeneities:

$$K^*_{MT} = K_M + c_I \frac{(K_I - K_M)K_M}{K_M + \alpha(1 - c_I)(K_I - K_M)}$$

$$\mu^*_{MT} = \mu_M + c_I \frac{(\mu_I - \mu_M)\mu_M}{\mu_M + \beta(1 - c_I)(\mu_I - \mu_M)}$$

$\alpha = \dfrac{3K_M}{3K_M + 4\mu_M}$

$\beta = \dfrac{6(K_M + 2\mu_M)}{5(3K_M + 4\mu_M)}$

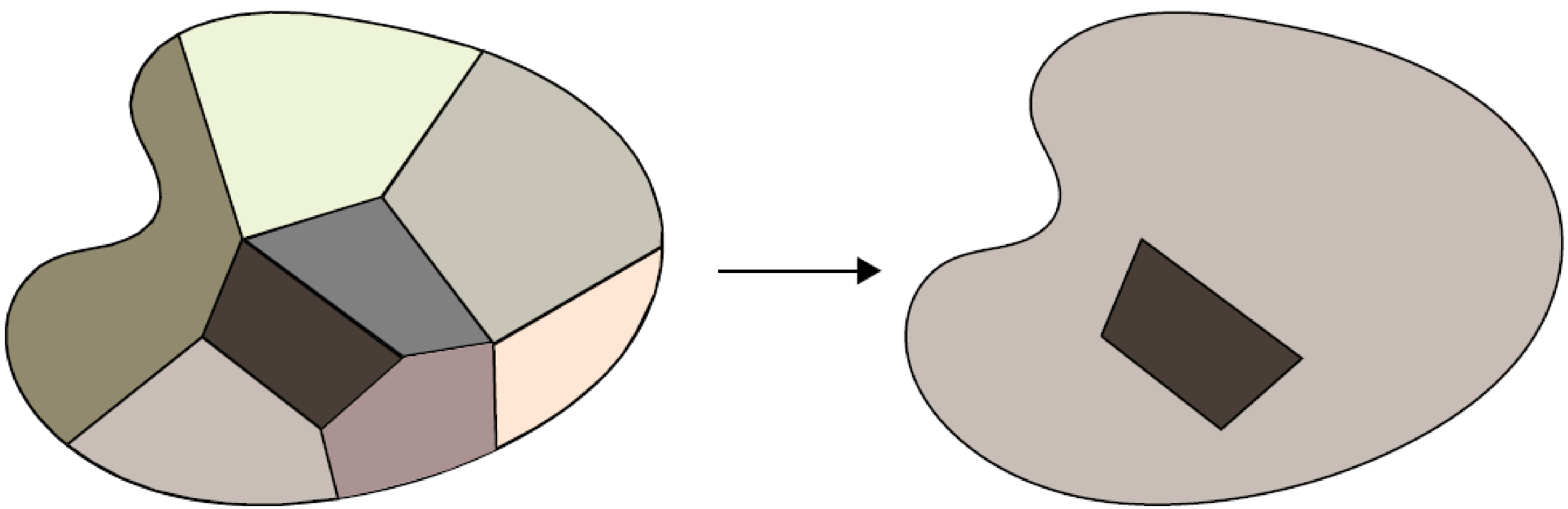
where α, β depend on K_M, μ_M and take the similar formular as in the DD approach. It is worthwhile to mention that K^*_{MT}, μ^*_{MT} do not depend linearly on c_I, in contrast to the DD method.

Particularly for hard particles ($K_I \to \infty, \mu_I \to \infty$) in incompressible matrix ($K_M \to \infty, \mu_M$), it holds $\alpha \to 1$ and $\beta \to \frac{2}{5}$. From the last equations,

$$\mu^*_{MT} \to (1 + \frac{5}{2}\frac{c_I}{1 - c_I})\mu_M, \qquad K^*_{MT} \to \infty$$

§ 4.7 Self-consistent approach (SC)

Both the DD and MT methods are suitable for small volume concentration of isolated defects. In cases like polycrystals, where the matrix phase and inhomogeneities are difficult to differentiate, both the DD and the MT approach is inappropriate. The basic idea of the self-consistent method is to smear the whole environment around the defect to be an effective matrix.



The SC method holds also for cases, where there is no distinct matrix phase. The formula in the SC method are similar to those in the DD method, except that in the effective problem the matrix with $\mathbf{E}_M$ is replaced by the unknown effective stiffness $\mathbf{E}^*$.

$$E^* = E_M + c_I(E_I - E_M)A_I$$

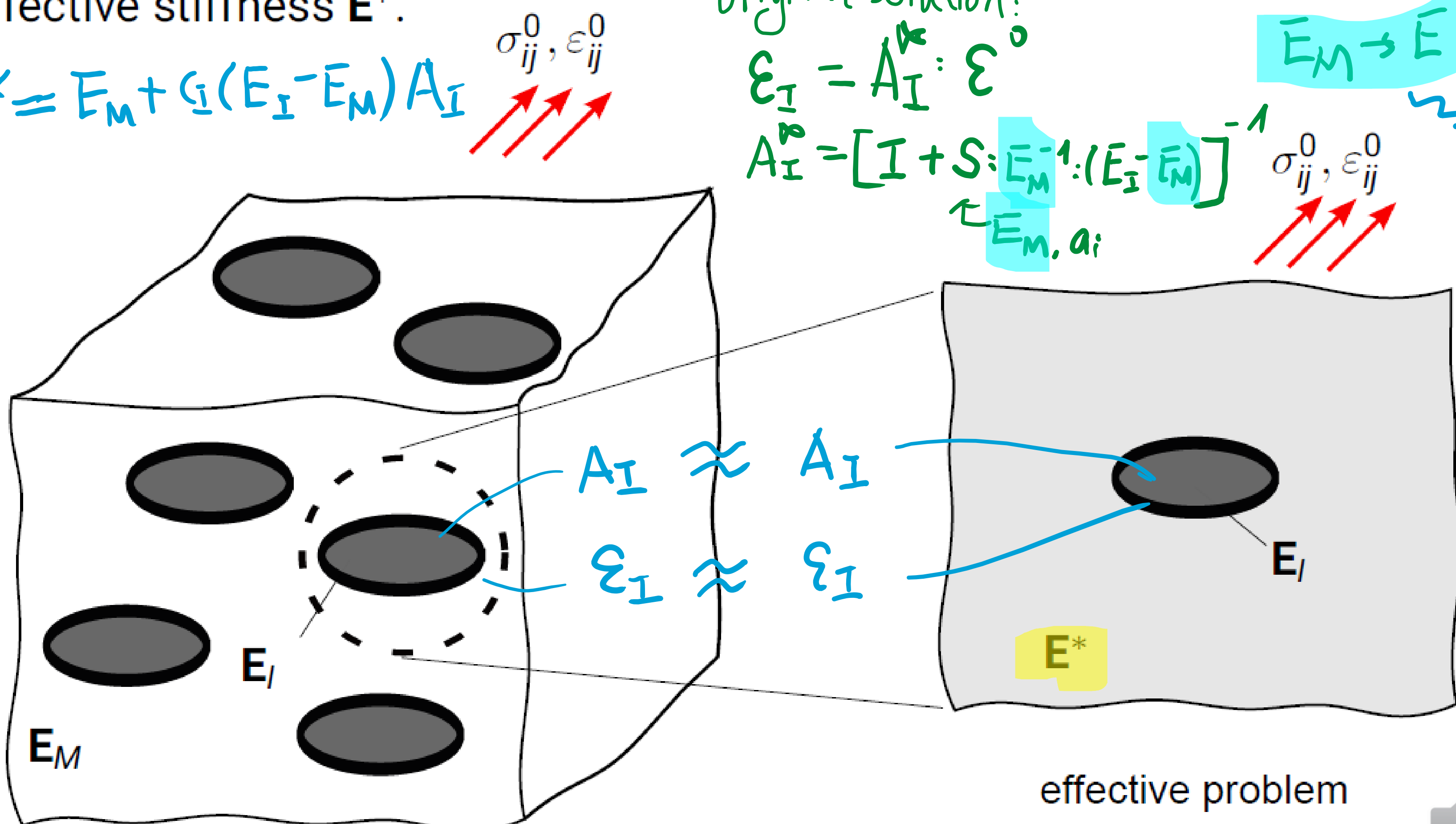
original solution:

$$\varepsilon_I = A_I^\infty : \varepsilon^0$$

$$A_I^\infty = [I + S : E_M^{-1} : (E_I - E_M)]^{-1}$$

$\leftarrow E_M, a_i$

$E_M \to E^*$ $\rightsquigarrow A_I^{SC}$



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The only difference from the DD approach lies in the fact that in the influence tensor $\mathbf{A}_{DD}$ the tensor $\mathbf{E}_M$ must be replaced by $\mathbf{E}^*$:

$$\boxed{\mathbf{A}_I^{SC} = [\mathbf{I} + \mathbf{S}^* : (\mathbf{E}^*)^{-1} : (\mathbf{E}_I - \mathbf{E}^*)]^{-1}}$$

where $\mathbf{S}^*$ has the similar structure as $\mathbf{S}$ in DD does, but depend on $\mathbf{E}^*$ (instead of $\mathbf{E}_M$), because in the effective problem, the stiffness tensor of the matrix phase is $\mathbf{E}^*$.
Note that $\mathbf{A}_I^{SC}$ is a function of $\mathbf{E}^*$ and thus of $\mathbf{E}^*_{SC}$. Replacement into the general formula leads to the governing equation for the unknown $\mathbf{E}^*_{SC}$:

$$\boxed{\mathbf{E}^*_{SC} = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I^{SC}} \tag{4.6}$$

$$\mathbf{E}^*_{SC} = \mathbf{E}_M + c_I(\mathbf{E}_I - \mathbf{E}_M) : \mathbf{A}_I^{SC}$$

$$\mathbf{A}_I^{SC} = [\mathbf{I} + \mathbf{S}^* : (\mathbf{E}^*)^{-1} : (\mathbf{E}_I - \mathbf{E}^*)]^{-1}$$

$\mathbf{E}_M \hat{=} (3K_M, 2\mu_M)$

$\mathbf{E}_I \hat{=} (3K_I, 2\mu_I)$

$\mathbf{E}^*_{SC} \hat{=} (3K^*_{SC}, 2\mu^*_{SC})$

$\mathbf{S}^* \hat{=} (\alpha^*, \beta^*)$

Comments:

- Isotropic case: $\mathbf{E}_I$, $\mathbf{E}_M$ are isotropic and spherical inhomogeneity

  One obtains first $\mathbf{S}^*$ through insertion $K_M \to K^*_{SC}, \mu_M \to \mu^*_{SC}$ into $\mathbf{S}$,

$$S^*_{ijkl} = \alpha^* \frac{1}{3}\delta_{ij}\delta_{kl} + \beta^*(I_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl}), \quad \alpha^* = \frac{3K^*_{SC}}{3K^*_{SC} + 4\mu^*_{SC}}, \ \beta^* = \frac{6(K^*_{SC} + 2\mu^*_{SC})}{5(3K^*_{SC} + 4\mu^*_{SC})}$$

$$\cancel{3}K^* = \cancel{3}K_M + \frac{c_I(\cancel{3}K_I - \cancel{3}K_M)}{1 + \alpha^* \cdot \frac{1}{\cancel{3}K^*}(\cancel{3}K_I - \cancel{3}K^*)} \rightsquigarrow \frac{\alpha^*}{K^*} = \left[\frac{c_I(K_I - K_M)}{K^* - K_M} - 1\right] \cdot \frac{1}{K_I - K^*}$$

$$\frac{\alpha^*}{K^*} = \frac{c_I}{K^* - K_M} \frac{K_I - K_M}{K_I - K^*} + \frac{1}{K^* - K_I} = \frac{c_I}{K^* - K_M} - \frac{c_I}{K^* - K_I} + \frac{1}{K^* - K_I} = \frac{c_I}{K^* - K_M} + \frac{c_M}{K^* - K_I}$$

$$\frac{c_I}{K^* - K_M} + \frac{c_M}{K^* - K_I} = \frac{3}{3K^* + 4\mu^*}$$

Due to isotropy, Eq. (4.6) leads to

$$0 = \frac{c_M}{K^*_{SC} - K_I} + \frac{c_I}{K^*_{SC} - K_M} - \frac{3}{3K^*_{SC} + 4\mu^*_{SC}}$$

$$0 = \frac{c_M}{\mu^*_{SC} - \mu_I} + \frac{c_I}{\mu^*_{SC} - \mu_M} - \frac{6(K^*_{SC} + 2\mu^*_{SC})}{5\mu^*_{SC}(3K^*_{SC} + 4\mu^*_{SC})}$$

The equation is symmetric with respect to the matrix and the inhomogeneity. In other words, a distinct matrix phase is not required.
Particularly for hard particles ($K_I \to \infty, \mu_I \to \infty$) and incompressible matrix ($K_M \to \infty, \mu_M$), one has

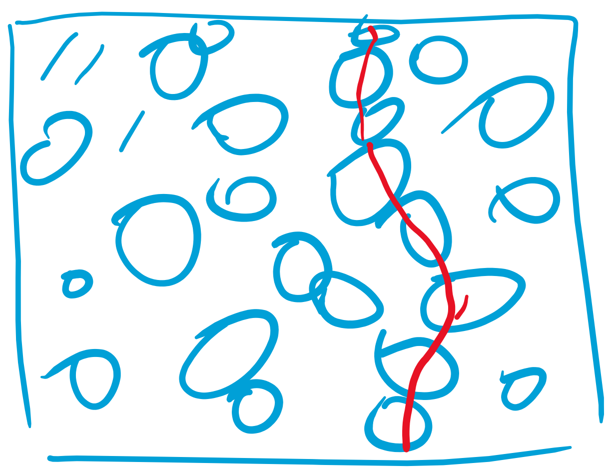
$$\mu^*_{SC} \to \frac{2\mu_M}{2 - 5c_I}$$

Even when $c_I \to \dfrac{2}{5}$ it holds already $\mu^*_{SC} \to \infty (= \mu_I)$. This is the so-called "Percolation effect". In the reality the microstructure forms already bridges even before $c_I < 1$.

Another special case is spherical cavities ($K_I \to 0$, $\mu_I \to 0$) in incompressible matrix ($K_M \to \infty$, μ_M). The effective constants, according to the SC approach are

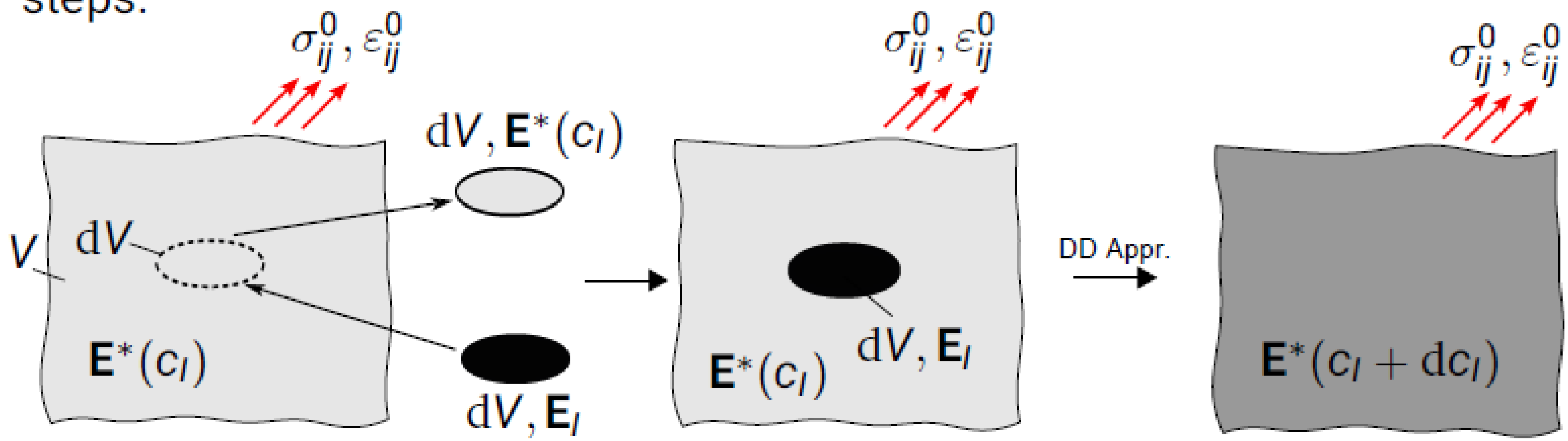
$$K^*_{SC} = \frac{4\mu_M(1-2c_I)(1-c_I)}{c_I(3-c_I)}, \qquad \mu^*_{SC} = \frac{3\mu_M(1-2c_I)}{3-c_I}$$

For $c_I = \frac{1}{2}, K^*_{SC} \to 0 (= K_I)$. Hereby the percolation effect appears.



§ 4.8 Differential scheme (DS)

Basic idea: Embedding the inhomogeneities into the matrix by infinitesimal steps.



$$c_I V - c_I \mathrm{d}V + \mathrm{d}V = (c_I + \mathrm{d}c_I)V$$

$$\rightarrow \quad \mathrm{d}V(1 - c_I) = V\mathrm{d}c_I \quad \rightarrow \quad \frac{\mathrm{d}V}{V} = \frac{\mathrm{d}c_I}{1 - c_I}$$

Matrix: $\mathbf{E}^*(c_I)$; Inhomogeneity: $\frac{dV}{V}, \mathbf{E}_I$

$\rightarrow$ $\mathbf{E}^*(c_I + dc_I)$ according to the DD approach.

E_M, a_i

It follows that $E^*_{DD} = E_M + c_I (E_I - E_M) : [I + S : E_M^{-1} : (E_I - E_M)]^{-1}$

$$\mathbf{E}^*(c_I + dc_I) = \mathbf{E}^*(c_I) + \frac{dV}{V}(\mathbf{E}_I - \mathbf{E}^*(c_I)) : [\mathbf{I} + \mathbf{S}^* : (\mathbf{E}^*(c_I))^{-1} : (\mathbf{E}_I - \mathbf{E}^*(c_I))]^{-1}$$

$\uparrow E^*(c_I), a_i$

where $\mathbf{S}^*$ depend on $\mathbf{E}^*(c_I)$ and the geometry of ellipsoids.

We denote the change of the stiffness tensor by $d\mathbf{E}^*$, and apply the relation $\frac{dV}{V} = \frac{dc_I}{1 - c_I}$. The last equations lead to

$= E^*(c_I + dc_I) - E^*(c_I)$

$$d\mathbf{E}^* = \frac{dc_I}{1 - c_I}(\mathbf{E}_I - \mathbf{E}^*(c_I)) : [\mathbf{I} + \mathbf{S}^* : (\mathbf{E}^*(c_I))^{-1} : (\mathbf{E}_I - \mathbf{E}^*(c_I))]^{-1}$$

or

$$\boxed{\frac{d\mathbf{E}^*}{dc_I} = \frac{1}{1 - c_I}(\mathbf{E}_I - \mathbf{E}^*(c_I)) : [\mathbf{I} + \mathbf{S}^* : (\mathbf{E}^*(c_I))^{-1} : (\mathbf{E}_I - \mathbf{E}^*(c_I))]^{-1}} \quad (4.7)$$

Comments:

- The last equation represents the nonlinear differential equation for $\mathbf{E}^*$ and c_I.
- The stiffness tensor of the real matrix with $\mathbf{E}_M$ appears not directly in the differential equation. But it appears in the boundary condition of the differential equation

$$\mathbf{E}^*|_{c_I=0} = \mathbf{E}_M$$

- It should be commented that in general the DS approach holds for the extreme cases:

$$c_I \to 0 : \qquad \mathbf{E}^*_{DS} = \mathbf{E}_M$$

$$c_I \to 1 : \qquad \mathbf{E}^*_{DS} = \mathbf{E}_I$$

- The DS concept is difficult to be applied for multiphase materials.

- For the special case of isotropic $\mathbf{E}_I, \mathbf{E}_M$ and spherical $\mathrm{d}V$, one obtains first $\mathbf{S}^*$ by replacing $K_M \to K^*(c_I), \mu_M \to \mu^*(c_I)$ in $\mathbf{S}$,

$$S^*_{ijkl} = \alpha^* \frac{1}{3}\delta_{ij}\delta_{kl} + \beta^*(I_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl})$$

$$\alpha^* = \frac{3K^*(c_I)}{3K^*(c_I) + 4\mu^*(c_I)}, \quad \beta^* = \frac{6(K^*(c_I) + 2\mu^*(c_I))}{5(3K^*(c_I) + 4\mu^*(c_I))}$$

For the particular case of hard particles ($K_I \to \infty, \mu_I \to \infty$) and incompressible matrix ($K_M \to \infty, \mu_M$), one has

$$\frac{\mathrm{d}\mu^*}{\mathrm{d}c_I} = \frac{1}{1-c_I}\frac{5}{2}\mu^*, \qquad \mu^*|_{c_I=0} = \mu_M$$

$$\to \quad \mu^* = \frac{\mu_M}{(1-c_I)^{5/2}}$$

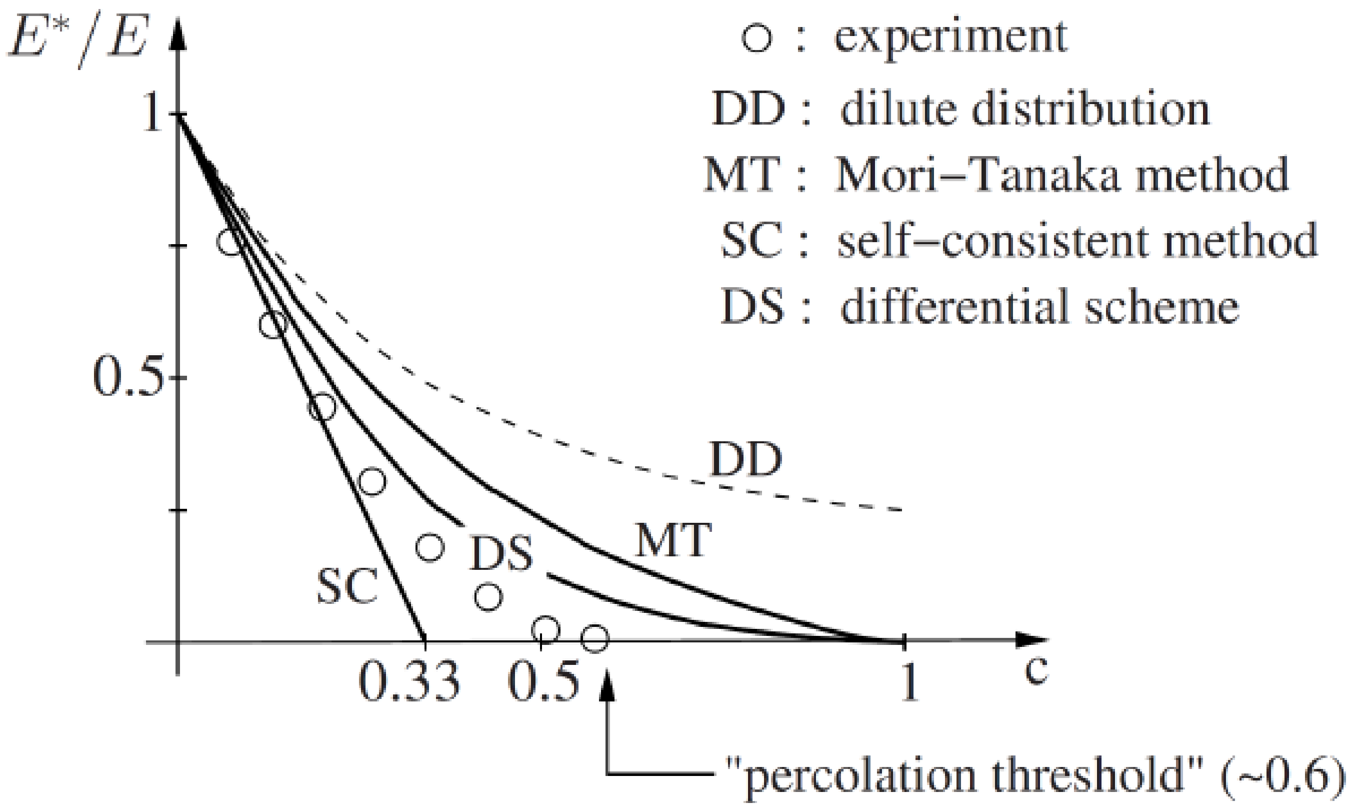
Note that for $c_I \to 1$, $\mu^* \to \infty$.

$\frac{d\mu^*}{dc_I} = f(K^*, \mu^*)$ $\quad K^*|_{c_I=0} = K_M$

$\frac{dK^*}{dc_I} = g(K^*, \mu^*)$ $\quad \mu^*|_{c_I=0} = \mu_M$

$$\boxed{\frac{\mathrm{d}\mathbf{E}^*}{\mathrm{d}c_I} = \frac{1}{1-c_I}(\mathbf{E}_I - \mathbf{E}^*(c_I)) : [\mathbf{I} + \mathbf{S}^* : (\mathbf{E}^*(c_I))^{-1} : (\mathbf{E}_I - \mathbf{E}^*(c_I))]^{-1}}$$

Effective Young's modulus of a plate containing isotropically distributed circular holes



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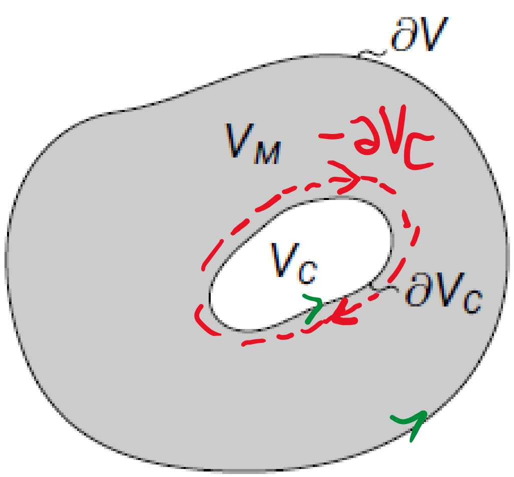
Homogeneous matrix with cracks and holes

$E^* = \langle E : A \rangle \qquad E^{*-1} = \langle E^{-1} : B \rangle$

$$< \varepsilon_{ij} >_M = \frac{1}{2V_M} \int_{V_M} (u_{i,j} + u_{j,i}) \mathrm{d}V = \frac{1}{2V_M} \left[\int_{\partial V} (u_i n_j + u_j n_i) \mathrm{d}A - \int_{\partial V_C} (u_i n_j + u_j n_i) \mathrm{d}A \right]$$

$$\langle \varepsilon_{ij} \rangle = \frac{1}{2V} \int_{\partial V} (u_i n_j + u_j n_i) \, dA$$



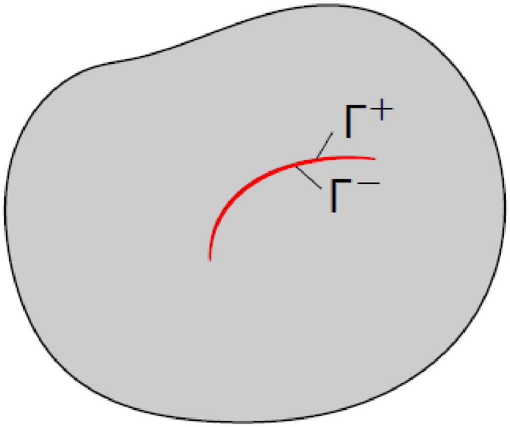
From Eq. (4.1) one has $\int_{\partial V} (u_i n_j + u_j n_i) \mathrm{d}A = 2V < \varepsilon_{ij} >$.

The last equations lead to

$$\rightarrow \quad < \varepsilon_{ij} >_M = \frac{1}{2V_M} \left[2V < \varepsilon_{ij} > - \int_{\partial V_C} (u_i n_j + u_j n_i) \mathrm{d}A \right]$$

$$\rightarrow \quad < \varepsilon_{ij} > = c_M < \varepsilon_{ij} >_M + \underbrace{\frac{1}{2V} \int_{\partial V_C} (u_i n_j + u_j n_i) \mathrm{d}A}_{\langle \varepsilon_{ij} \rangle_C}$$

MECHANICS of FUNCTIONAL MATERIALS



For matrix with cracks, $\partial V_C \rightarrow \Gamma = \Gamma^+ + \Gamma^-$ und $c_M = 1$. One obtains from the last equations

$$< \varepsilon_{ij} > = < \varepsilon_{ij} >_M + \underbrace{\frac{1}{2V}\int_{\Gamma^+} (\Delta u_i n_j + \Delta u_j n_i)\mathrm{d}A}_{< \varepsilon_{ij} >_C}$$

where $\Delta u_i = u_i^+ - u_i^-$ denotes the displacement discontinuities along the crack surfaces.

In summary,

$$\boxed{< \varepsilon_{ij} > = c_M < \varepsilon_{ij} >_M + < \varepsilon_{ij} >_C \quad \text{or} \quad c_M < \varepsilon_{ij} >_M = < \varepsilon_{ij} > - < \varepsilon_{ij} >_C}$$

$$\boxed{< \varepsilon_{ij} >_C = \begin{cases} \dfrac{1}{2V}\displaystyle\int_{\partial V_C} (u_i n_j + u_j n_i)\mathrm{d}A & \text{for cavities with boundary } \partial V_C \\ \dfrac{1}{2V}\displaystyle\int_{\Gamma^+} (\Delta u_i n_j + \Delta u_j n_i)\mathrm{d}A & \text{for cracks with boundary } \Gamma \end{cases}} \quad (4.8)$$

The mean values of the stresses in a matrix with cracks and holes can be further considered. Since no stress exists in cracks,

$$<\boldsymbol{\sigma}>= c_M <\boldsymbol{\sigma}>_M$$

Particularly for $\mathbf{E}_M = const.$ one has $<\boldsymbol{\sigma}>_M = \mathbf{E}_M :<\varepsilon>_M$. It follows that

$$<\boldsymbol{\sigma}>= c_M \mathbf{E}_M :<\varepsilon>_M$$

Insertion of $c_M <\varepsilon_{ij}>_M = <\varepsilon_{ij}> - <\varepsilon_{ij}>_C$ leads to

$$\boxed{<\boldsymbol{\sigma}>= \mathbf{E}_M : (<\varepsilon> - <\varepsilon>_C) \quad \text{or} \quad <\varepsilon>= \mathbf{E}_M^{-1} :<\boldsymbol{\sigma}> + <\varepsilon>_C}$$

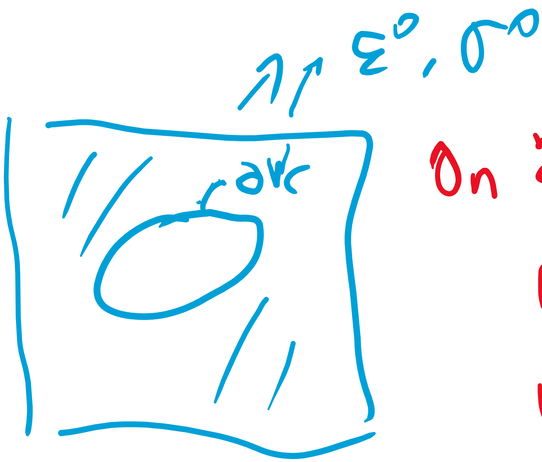
$\sim <\varepsilon>$ $\sim <\sigma>$

In the case of linear elasticity, one can assume that the displacement along the crack and on the hole surfaces are proportional to the loading on the RVE, and thus to the average strain $<\varepsilon>$ or $<\boldsymbol{\sigma}>$. Thus

a) $$<\varepsilon>_C = \mathbf{D} : \varepsilon^0 = \mathbf{D} :<\varepsilon>$$

b) $$<\varepsilon>_C = \mathbf{H} : \boldsymbol{\sigma}^0 = \mathbf{H} :<\boldsymbol{\sigma}>$$

where **D** and **H** are kind of influence tensors (4th order), which depend only on the matrix material and the geometry of the defects. Through **D** and **H** one has the following general expressions for the effective elastic properties of the material with cracks or holes:



$$\langle \varepsilon_{ij} \rangle_c = \frac{1}{2V} \int_{\partial V_c} (u_i n_j + u_j n_i)\, dA$$

On ∂V_c,

$u \sim \varepsilon^0, \langle\varepsilon\rangle$, $\langle\varepsilon\rangle_c \sim \varepsilon^0, \langle\varepsilon\rangle$

$u \sim \sigma^0, \langle\sigma\rangle$, $\langle\varepsilon\rangle_c \sim \sigma^0, \langle\sigma\rangle$

BC a)

$\langle \varepsilon \rangle_c = \mathbf{D} : \langle \varepsilon \rangle$

$$<\boldsymbol{\sigma}> = \mathbf{E}_M : (<\varepsilon> - <\varepsilon>_C) = \mathbf{E}_M : (\mathbf{I} - \mathbf{D}) :<\varepsilon>$$

$$\rightarrow \qquad \boxed{\mathbf{E}^* = \mathbf{E}_M : (\mathbf{I} - \mathbf{D})}$$

It is notable, that **D** indicates the decrease of stiffness due to the presence of cracks and holes (Damage).

BC b)

$\langle \varepsilon \rangle_c = \mathbf{H} : \langle \sigma \rangle$

$$<\varepsilon> = \mathbf{E}_M^{-1} :<\boldsymbol{\sigma}> + <\varepsilon>_C = \underbrace{\left(\mathbf{E}_M^{-1} + \mathbf{H}\right)}_{(\mathbf{E}^*)^{-1}} :<\boldsymbol{\sigma}>$$

$$\rightarrow \qquad \boxed{\mathbf{E}^* = \left(\mathbf{E}_M^{-1} + \mathbf{H}\right)^{-1}}$$

The tensor **H** implies the increase of the compliance due to the existence of cracks and holes (Softening).

Using the various approximation methods, such as DD, MT, SC, DA methods, one can determine also **H** and **D**, respectively, for each crack and hole. This leads to approximation of the effective properties of the material.