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INDEX SETS IN THE ARITHMETICAL HIERARCHY

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We prove the following results: every recursively enumerable set approximated by finite sets of some set M of recursively enumerable sets with index set in Π_2 is an element of M, provided that the finite sets in M are canonically enumerable. If both the finite sets in M and in $\bar{M}$ are canonically enumerable, then the index set of M is in $\Sigma_2 \cap \Pi_2$ if and only if M consists exactly of the sets approximated by finite sets of M and the complement $\bar{M}$ consists exactly of the sets approximated by finite sets of $\bar{M}$. Under the same condition M or $\bar{M}$ has a non-empty subset with recursively enumerable index set, if the index set of M is in $\Sigma_2 \cap \Pi_2$.

If the finite sets in M are canonically enumerable, then the following three statements are equivalent: (i) the index set of M is in $\Sigma_2 \backslash \Pi_2$, (ii) the index set of M is Σ_2-complete, (iii) the index set of M is in Σ_2 and some sequence of finite sets in M approximate a set in $\bar{M}$.

Finally, for every $n \geq 2$, an index set in $\Sigma_n \backslash \Pi_n$ is presented which is not Σ_n-complete.

Introduction

The well known theorems of Rice and Rice–Shapiro [9] characterize sets of indices (index sets) of sets of partial recursive (p.r.) functions located in Σ_0 and Σ_1. An index set is the set of all 'programs' computing the functions of the given set. Up to now there has been no such characterization for higher steps in the arithmetical hierarchy. The interest in studying index sets located on higher steps of the hierarchy—especially between $\Sigma_3 \cap \Pi_3$ and $\Sigma_2 \cap \Pi_2$—is motivated by results related to the inductive inference problem. It can be shown that any identifiable function set is included in an identifiable function set with index set in $\Sigma_3 \cap \Pi_3$ [2], [3]. Thus, there is an obvious desire to get more informations about function sets with index sets in $\Sigma_3 \cap \Pi_3$. None of the identifiable sets can include the whole set of recursive functions (Gold [4]). Moreover with the help of Gold's result and the Rice–Shapiro theorem it can be shown that no identifiable set can include a non-empty subset with index set in Σ_1; hence no non-empty function set with index set in Σ_1 is identifiable [3].

We want to present three results on this topic: every recursively enumerable (r.e.) set approximated by finite sets of some set M of r.e. sets with index set in Π_2 is an element of M, provided that the finite sets in M are canonically enumerable. If both the finite sets in M and in $\bar{M}$ are canonically enumerable, then the index set of M is in $\Sigma_2 \cap \Pi_2$ if and only if M consists exactly of the sets approximated by finite sets of M and the complement $\bar{M}$ consists exactly of the sets approximated by finite sets of $\bar{M}$. Under the same condition M or $\bar{M}$ has a non-empty subset with r.e. index set, if the index set of M is in $\Sigma_2 \cap \Pi_2$.

Furthermore we investigate Σ_2-completeness of index sets. The relation between intuitive simplicity of definition and completeness or non-completeness of arithmetical sets is a problem not fully understood [9, p. 330]. Almost all index sets studied in the past have been proved to be Σ_n-complete or Π_n-complete. The question of completeness of index sets has been further discussed by D. E. Miller. He shows in [8] that every naturally defined class of sets includes an index set which is 1-complete for that class and that every index set is 1-complete for some naturally defined class, where 'naturally defined' is formalized as 'effective Boolean'.

According to Rice's theorem (or, more precisely, its proof as given in Rogers [9]) every non-trivial index set in Σ_1 is Σ_1-complete. Assuming that the finite sets in M are canonically r.e., we shall show, that the following three statements are equivalent:

(i) The index set of M is in $\Sigma_2 \backslash \Pi_2$.

(ii) The index set of M is Σ_2-complete.

(iii) The index set of M is in Σ_2 and some sequence of finite sets in M approximate a set in $\bar{M}$.

If we drop the assumption that the finite sets in M are canonically r.e., then such a characterization seems to be rather difficult. This is indicated by the fact that there are index sets in $\Sigma_2 \backslash \Pi_2$ which are not complete on that level of the arithmetical hierarchy. Using the result of Yates [10] that $\{z \mid W_z \equiv_{\mathrm{T}} A\}$ is Σ_3^A-complete, Rogers [9] presents an index set in $\Sigma_4 \backslash \Pi_4$ which is not Σ_4-complete, choosing an appropriate A. This example is easily modified to provide for all $n > 1$ examples of index sets in $\Sigma_n \backslash \Pi_n$ which are not Σ_n-complete: let A be the well-known example (due to Lachlan and Sacks, see e.g. Theorem 13-XXVI of [9]) of an r.e. set satisfying $\emptyset^{(n)} <_{\mathrm{T}} A^{(n)} <_{\mathrm{T}} \emptyset^{(n+1)}$ for all n. As remarked in Theorem 5.2 of [5] $\{i \mid W_i \cap B' \neq \emptyset\}$ is Σ_1^B-complete for all sets B; hence for all $n \geq 1$, $C_n = \{i \mid W_i \cap A^{(n)} \neq \emptyset\} \equiv_m A^{(n)}$ is an example of an index set which is in $\Sigma_{n+1} \backslash \Pi_{n+1}$ but not Σ_{n+1}-complete.

In the last part of this paper we provide other examples of index sets on every level >1 of the arithmetical hierarchy which are not complete on that level. The examples are 'complementary' to the examples above in that they have the form $\{i \mid W_i \subseteq S\}$ while the above examples have form $\{i \mid W_i \nsubseteq S\}$.

Basic notations and definitions

We assume that the reader is familiar with the basic concepts and results of recursion theory [9]. We adopt the notations of [9]; in particular an acceptable enumeration of the unary p.r. functions is denoted by $(\phi_i)_{i=0}^{\infty}$. For any given ϕ_i, W_i

is the domain of ϕ_i ($\mathrm{dom}(\phi_i) = W_i$). W_i^n is the set of all x (with respect to a fixed dovetailing procedure) occurring within the first n steps of the enumeration of W_i (depending on the index i). The notation transfers to the relativized theory by indexing with the oracle (ϕ_i^B, W_i^B, $W_i^{B,n}$). If M is a set of r.e. sets, then $\mathrm{Ind}(M) = \{i \mid \phi_i \in M\}$ is the index set of M.

The finite set with canonical index x is D_x [9]. Suppose a set M of r.e. sets is given. The finite sets in M are *canonically enumerable* (c.e.) if and only if there is a recursive function f with $\{D_{f(x)} \mid x \in \mathbb{N}\} = \{W \mid W \in M \text{ and } W \text{ finite}\}$. M is called *canonical* if and only if the finite sets in M are canonically enumerable.

The next definition deals with the concept of approximating r.e. sets by finite sets. A r.e. set A is called *approximable* by finite sets in M if and only if for every $D_x \subseteq A$ there exists a $D_y \in M$ with $D_x \subseteq D_y \subseteq A$. By Approx($M$) we denote the set of all r.e. sets A which are approximable by finite sets in M. According to the definition any finite set is in M if and only if it is an element of Approx(M).

Consider for example the set $M = \{J \mid \exists n \in \mathbb{N} : J = [0:n]\}$. Then there exists exactly one infinite set approximated by finite sets in M, namely the set $\mathbb{N}$. We get $\mathrm{Approx}(M) = M \cup \{\mathbb{N}\}$.

In the following let E denote the set of all finite subsets of $\mathbb{N}$.

Index sets in Σ_2 and $\Sigma_2 \cap \Pi_2$

We fix a set M of r.e. sets. We want to show first that Approx(M) is a subset of M, provided that M is canonical and $\mathrm{Ind}(M) \in \Pi_2$.

Example. Consider $M = \{A \mid A \text{ r.e.} \wedge ((x = \mu y : y \in A) \Rightarrow D_x \subseteq A)\}$. M includes every nonempty r.e. set A whose minimal element is a canonical index for a finite set included in A, therefore

$$\begin{aligned} M &= \{A \mid A \text{ r.e.} \wedge \forall x, y\, [(x \in A \wedge (y < x \Rightarrow y \notin A)) \Rightarrow D_x \subseteq A]\} \\ &= \{A \text{ r.e.} \mid \exists x\, (x \in A \wedge D_x \subseteq A \wedge \forall y\, (y \in A \Rightarrow y \geq x)) \vee \forall x\, (x \notin A)\}. \end{aligned}$$

With the help of the Tarski–Kuratowski algorithm [9] we get

$$\begin{aligned} &\mathrm{Ind}(M) \\ &\quad = \{i \mid \forall x, y\, [(\exists n\, (x \in W_i^n \wedge (y < x \Rightarrow \forall m\, (y \notin W_i^m)))) \Rightarrow \exists l\, (D_x \subseteq W_i^l)]\} \\ &\quad = \{i \mid \forall x, y, n\, \exists m, l\, [(x \in W_i^n \wedge (y < x \Rightarrow y \notin W_i^m)) \Rightarrow D_x \subseteq W_i^l]\} \end{aligned}$$

as well as

$$\begin{aligned} \mathrm{Ind}(M) = \{i \mid \exists x, n, l\, \forall y, m, k \\ [(x \in W_i^n \wedge D_x \subseteq W_i^l \wedge (y \in W_i^m \Rightarrow y \geq x)) \vee W_i^k = \emptyset]\}. \end{aligned}$$

Hence $\mathrm{Ind}(M) \in \Sigma_2 \cap \Pi_2$.

The finite sets in M are c.e., even recursive since there are algorithms which compute for any x the cardinality of D_x and the elements of D_x and decide for any x and y whether D_x is contained in D_y or not, hence the characteristic function

$$\chi(x)=\begin{cases}1, & \text{if } (y=\mu z: z\in D_x)\Rightarrow D_y\subseteq D_x\\ 0, & \text{otherwise}\end{cases},$$

is recursive. Therefore there exists recursive functions f and h with $f(\mathbb{N})=\chi^{-1}(1)$ and $h(\mathbb{N})=\chi^{-1}(0)$. By this

$$\{D_{f(x)} \mid x\in\mathbb{N}\}=M\cap E \quad\text{and}\quad \{D_{h(x)} \mid x\in\mathbb{N}\}=\bar{M}\cap E,$$

i.e., M and $\bar{M}$ are both canonical.

We shall prove a lemma asserting that $\text{Approx}(M)\subseteq M$. Consider again the example and a set $A\in\bar{M}$ with minimal element z. By definition $D_z\nsubseteq A$, i.e., $D_z\nsubseteq D_y$ for every $D_y\subseteq A$. Hence we can state for every $D_y\subseteq A$ containing the element z that z is the minimal element in D_y and $D_z\nsubseteq D_y$, which means $D_y\in\bar{M}$. Thus $A\notin \text{Approx}(M)$.

Lemma 1. *Every canonical set M has the following properties*:
(i) *If* $\text{Approx}(M)\nsubseteq M$, *then* $\text{Ind}(E)\leqslant_m \text{Ind}(M)$.
(ii) *If* $M\nsubseteq \text{Approx}(M)$, *then* $\text{Ind}(E)\leqslant_m \text{Ind}(\bar{M})$.

Proof. Consider a canonical set M and a recursive function f enumerating the finite sets of M.

(i) If $\text{Approx}(M)\nsubseteq M$, there exists an $A\in\text{Approx}(M)\setminus M$ such that A is infinite since $\text{Approx}(M)\cap E\subseteq M$. Fix a recursive function g with $g(\mathbb{N})=A$. Define

$$A^{(0)}=\{g(0)\},\qquad A^{(n+1)}=A^{(n)}\cup\{g(n+1)\}$$

and

$$p(0)=f(\mu y:\exists k\,(A^{(0)}\subseteq D_{f(y)}\subseteq A^{(k)})),$$
$$p(n+1)=f(\mu y:\exists k\,(A^{(n)}\cup D_{p(n)}\subseteq D_{f(y)}\subseteq A^{(k)})).$$

Then p is recursive, $D_{p(n)}\subseteq D_{p(n+1)}\subseteq A$ for all n, and $\exists n\,(x\in D_{p(n)})$ iff $x\in A$; hence $\bigcup_n D_{p(n)}=A$. Define by the s^m_n-theorem a recursive function $\beta(i)$ to be the index of an r.e. set defined by

$$W_{\beta(i)}=\bigcup_{\exists z\geqslant y\,(z\in W_i)} D_{p(y)}=\{x\mid \exists y\,\exists z\geqslant y\,(z\in W_i\wedge x\in D_{p(y)})\}.$$

If W_i is finite and $m=\max\{y\mid y\in W_i\}$, then $W_{\beta(i)}=D_{p(m)}\in M$. If W_i is infinite, then $W_{\beta(i)}=\bigcup_n D_{p(n)}=A\notin M$. Hence W_i is finite iff $\beta(i)\in\text{Ind}(M)$. In conclusion $\text{Ind}(E)\leqslant_m\text{Ind}(M)$.

(ii) If $M\nsubseteq\text{Approx}(M)$ there exists an $A\in M\setminus\text{Approx}(M)$, i.e., there is an $D_x\subseteq A$ such that $\forall y\,(D_x\subseteq D_{f(y)}\Rightarrow D_{y(z)}\nsubseteq A)$. By this every finite subset of A properly including D_x is an element of $\bar{M}$. As in the first part of the proof let $A^{(0)}=\{g(0)\}$, $A^{(n+1)}=A^{(n)}\cup\{g(n+1)\}$. Define by the s^m_n-theorem a recursive function $\beta(i)$ to be the index of an r.e. set defined by

$$W_{\beta(i)} = \bigcup_{n \in W_i} D_x \cup A^{(n)}.$$

If W_i is infinite, $W_{\beta(i)} = A \in M$. If W_i is finite, we get $D_x \subseteq W_{\beta(i)} \subseteq A$ where $W_{\beta(i)}$ is finite and hence $\in \bar{M}$. In conclusion $\text{Ind}(E) \leqslant_m \bar{M}$. □

As an immediate consequence of Lemma 1 we get

Lemma 2. *Every canonical set M has the following properties*:
(i) *If* $\text{Ind}(M) \in \Pi_2$, *then* $\text{Approx}(M) \subseteq M$.
(ii) *If* $\text{Ind}(M) \in \Sigma_2$, *then* $M \subseteq \text{Approx}(M)$.

Look at the running example, for which we know $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$. We have shown $\text{Approx}(M) \subseteq M$. Now consider $A \in M$ with minimal element z so that $D_z \subseteq A$. Now, any D_y with $\{z\} \cup D_z \subseteq D_y \subseteq A$ has to be an element of M since z is its minimal element and $D_z \subseteq D_y$. To every $D_x \subseteq A$ such a D_y containing D_x can be found (namely $D_x \cup \{z\} \cup D_z$). Thus $A \in \text{Approx}(M)$. In conclusion we get $\text{Approx}(M) = M$.

We can now characterize the sets with $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$ as those sets with $\text{Approx}(M) = M$ and $\text{Approx}(\bar{M}) = \bar{M}$ provided that M and $\bar{M}$ are canonical.

Lemma 3. *If M is canonical and* $M = \text{Approx}(M)$, *then* $\text{Ind}(M) \in \Pi_2$.

Proof. Choose a recursive function f enumerating the canonical indices of the finite sets in M. If

$$M = \text{Approx}(M) = \{A \mid A \text{ r.e.} \wedge \forall D_x \subseteq A\ \exists D_y \in M : D_x \subseteq D_y \subseteq A\}$$

we get

$$M = \{A \mid \text{r.e.} \wedge \forall x\ \exists y\ (D_x \subseteq A \Rightarrow D_x \subseteq D_{f(y)} \subseteq A)\}.$$

Thus

$$\begin{aligned}\text{Ind}(M) &= \{i \mid \forall x\ \exists y\ (D_x \subseteq W_i \Rightarrow D_x \subseteq D_{f(y)} \subseteq W_i)\} \\ &= \{i \mid \forall x, n\ \exists y, m\ (D_x \subseteq W_i^n \Rightarrow D_x \subseteq D_{f(y)} \subseteq W_i^m)\} \in \Pi_2. \qquad \square\end{aligned}$$

Theorem 4. *If M is a set of r.e. sets such that M and* $\bar{M}$ *are canonical, then* $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$ *if and only if* $M = \text{Approx}(M)$ *and* $\bar{M} = \text{Approx}(\bar{M})$.

Proof. The only-if-part follows by Lemma 2, while the if-part is an immediate consequence of Lemma 3. □

If we consider canonical sets M with the property $\text{Ind}(M) \in \Sigma_2 \cap \Pi_2$ and $\bar{M}$ canonical, then we can show with the help of the Rice–Shapiro theorem that there exists a set $L \neq \emptyset$ such that $L \subseteq M$ or $L \subseteq \bar{M}$, with $\mathrm{Ind}(L)$ r.e. In our running example we can choose

$$L = \{A \mid A \text{ r.e. } \wedge \{0\} \cup D_0 \subseteq A\}.$$

According to the definition of M, $L \subseteq M$. It is easily seen that $\mathrm{Ind}(L)$ is r.e., or equivalently $\mathrm{Ind}(L) \in \Sigma_1$.

A set L of sets is called a basic open set if $L = \{A \text{ r.e.} \mid D_x \subseteq A\}$ for some D_x (see also p. 357 of [6]).

Theorem 5. *If M and $\bar{M}$ are canonical and* $\mathrm{Ind}(M) \in \Sigma_2 \cap \Pi_2$, *then M or $\bar{M}$ includes a non-empty basic open set.*

Proof. Choose some recursive functions f and h enumerating the canonical indices of the finite sets in M and $\bar{M}$. Now we proceed by contradiction supposing that neither M nor $\bar{M}$ include a basic open set, i.e., for every x, $\{A \text{ r.e.} \mid D_x \subseteq A\} \nsubseteq M$ and $\{A \text{ r.e.} \mid D_x \subseteq A\} \nsubseteq \bar{M}$. Hence we can find for every $D_x \in M$ a $D_y \in \bar{M}$ with $D_x \subseteq D_y$. Otherwise there is an infinite r.e. $B \in \bar{M} \cap \mathrm{Approx}(M)$ — a contradiction to Theorem 4. By the same argument there is for every $D_y \in \bar{M}$ a $D_x \in M$ with $D_y \subseteq D_x$. The fact that M and $\bar{M}$ are canonical even allows us to compute for every $D_x \in M$ a $D_y \in \bar{M}$ with $D_x \subseteq D_y$ using an algorithm computing $y = \mu z(D_x \subseteq D_{f(z)})$. The same holds for $D_y \in \bar{M}$: If $x = \mu z(D_y \subseteq D_{h(z)})$ we obtain $D_y \subseteq D_x$ and $D_x \in M$. Now define a function p by

$$p(0) = \mu z(z \in f(\mathbb{N})),$$
$$p(2n+1) = h(\mu z(D_{p(2n)} \subseteq D_{h(z)})),$$
$$p(2n+2) = f(\mu z(D_{p(2n+1)} \subseteq D_{f(z)})).$$

p is recursive. Define $A = \bigcup_n D_{p(n)}$. Then A is an infinite r.e. set. By construction $A \in \mathrm{Approx}(M) \cap \mathrm{Approx}(\bar{M})$ — a contradiction to Theorem 4. □

For every basic open set L, $\mathrm{Ind}(L) \in \Sigma_1$. Thus we get the following

Corollary. *If M and $\bar{M}$ are canonical and* $\mathrm{Ind}(M) \in \Sigma_2 \cap \Pi_2$, *then there is a non-empty set L completely included in M or in $\bar{M}$ with* $\mathrm{Ind}(L) \in \Sigma_1$.

Σ_2-completeness of index sets

A set A is Σ_n^B-complete if $A \in \Sigma_n^B$ and $\forall C\, (C \in \Sigma_n^B \Rightarrow C \leqslant_1 A)$. Π_n^B-completeness is defined similarly. According to exercise 14-10 in the book of Rogers [9] every Σ_n^B-complete set forms a 1-degree which is also an m-degree. Hence we get an equivalent definition replacing one–one reducibility ($\leqslant_1$) by many–one reducibility ($\leqslant_m$).

Theorem 6. *If M is canonical then the following statements are equivalent*:

(i) $\mathrm{Ind}(M)$ *is* Σ_2-*complete.*

(ii) $\mathrm{Ind}(M) \in \Sigma_2 \backslash \Pi_2$.

(iii) $\mathrm{Ind}(M) \in \Sigma_2$ *and* $\mathrm{Approx}(M) \nsubseteq M$.

Proof. Clearly, (i) implies (ii).

If $\mathrm{Ind}(M) \in \Sigma_2 \backslash \Pi_2$, then $M \subseteq \mathrm{Approx}(M)$ by Lemma 2(ii) and therefore $\mathrm{Approx}(M) \nsubseteq M$ (otherwise $\mathrm{Ind}(M) \in \Pi_2$ by Lemma 3). Thus (ii) ⇒ (iii).

(iii) ⇒ (i) follows directly by Lemma 1(i). □

The topic of the last part of this paper is to present index sets in $\Sigma_2^B \backslash \Pi_2^B$ which are not Σ_2^B-complete. To do this we need some preparations. For every set M of r.e. sets $\mathrm{Ind}(M)_{\min} = \{i \mid W_i \in M \wedge \forall j < i\, (W_i \neq W_j)\}$ is the set of all minimal indices for sets in M.

The following result is due to M. Blum, probably unpublished. Reference is made to it in [7].

Lemma 7. *If M is infinite, then* $\mathrm{Ind}(M)_{\min}$ *is immune.*

Proof. According to Theorem 1 in [1] for every recursive function g with infinite range there exist i, j such that $\phi_i = \phi_{g(j)}$ and $i < g(j)$. Supposing that $\mathrm{Ind}(M)_{\min}$ is not immune we can choose g such that $g(\mathbb{N}) \subseteq \mathrm{Ind}(M)_{\min}$. But then $\phi_i = \phi_{g(j)}$ (i.e., $W_i = W_{g(j)}$) and $i < g(j)$ for some i and j means that $g(j) \notin \mathrm{Ind}(M)_{\min}$ — a contradiction. □

Lemma 8. *If* $\mathrm{Ind}(M) \in \Sigma_2^B \backslash \Pi_2^B$, *then*

(i) $\mathrm{Ind}(M)_{\min} \in \Sigma_2^B \backslash \Pi_2^B$, *and*

(ii) $\mathrm{Ind}(M)_{\min}$ *is immune.*

Proof. (i) The fact that $\mathrm{Ind}(M)_{\min} \in \Sigma_2^B$ follows directly from the assumption $\mathrm{Ind}(M) \in \Sigma_2^B$ by the Tarski–Kuratowski algorithm [9] since

$$i \in \mathrm{Ind}(M)_{\min} \quad \Leftrightarrow \quad i \in \mathrm{Ind}(M) \wedge \forall j < i\, (W_i \neq W_j)$$

where $i \in \mathrm{Ind}(M) \in \Sigma_2^B$ and

$$\forall j < i\, (W_i \neq W_j) \quad \Leftrightarrow \quad \forall j < i\, \exists x\, ((x \in W_i \wedge x \notin W_j) \vee (x \in W_j \wedge x \notin W_i))$$

which is in Σ_2 and hence in Σ_2^B by standard Tarski–Kuratowski manipulations.

Similarly

$$i \in \mathrm{Ind}(M) \quad \Leftrightarrow \quad \exists j \leqslant i\, (j \in \mathrm{Ind}(M)_{\min} \wedge W_i = W_j)$$

where

$$W_i = W_j \quad \Leftrightarrow \quad \forall x\,((x \in W_i \wedge x \in W_j) \vee (x \notin W_i \wedge x \notin W_j))$$

is in Π_2 and hence in Π_2^B.

If $\mathrm{Ind}(M)_{\min} \in \Pi_2^B$ as well, then $\mathrm{Ind}(M) \in \Pi_2^B$ by the Tarski–Kuratowski algorithm, contradicting the hypothesis that $\mathrm{Ind}(M) \in \Sigma_2^B \backslash \Pi_2^B$.

(ii) Observe that M must be infinite; otherwise $\mathrm{Ind}(M)_{\min}$ is finite contradicting the fact that $\mathrm{Ind}(M)_{\min} \in \Sigma_2^B \backslash \Pi_2^B$. Thus $\mathrm{Ind}(M)_{\min}$ is immune by Lemma 7. □

In the following let $B' = \{x \mid x \in W_x^B\}$ and $B'' = (B')'$. Furthermore we denote by $\mathrm{Ind}_B(M) = \{i \mid W_i^B \in M\}$ the *B-index set of M*. In the proof of the following two lemmata we use the fact that $\mathrm{Ind}_B(E) \equiv_m B''$ and $\mathrm{Ind}_B(\{\emptyset\}) \equiv_m \overline{B'}$. The two statements can be proved by replacing in the proofs for $\mathrm{Ind}(E) \equiv_m \emptyset''$ and $\mathrm{Ind}(\{\emptyset\}) \equiv_m \overline{\emptyset'}$ the partial recursive functions by the partial B-recursive functions. We skip the proof here. Finally we define for every set B, $S_B = \{W \mid W \text{ r.e. and } W \subseteq B\}$ as the set of all r.e. subsets of B and $FS_B = \{W \mid W \text{ finite and } W \subseteq B\}$ as the set of all finite subsets of B. It is easily verified that $B \leqslant_m \mathrm{Ind}(S_B)$ and $B \leqslant_m \mathrm{Ind}(FS_B)$.

Lemma 9. $\mathrm{Ind}(S_{B'}) \equiv_m \overline{B''}$.

Proof. This is an immediate consequence of the well-known fact that if S is Σ_n^B-complete, then $\{i \mid W_i \subseteq S\}$ is Π_{n+1}^B-complete (see e.g. the note after Theorem 6.3 of [5]). □

Lemma 10. *If* $C \leqslant_m B''$, *then* $\mathrm{Ind}(FS_C) \leqslant_m B''$.

Proof. By the above observation $C \leqslant_m B'' \equiv_m \mathrm{Ind}_B(E)$. Hence there exists a recursive function f with $f^{-1}(\mathrm{Ind}_B(E)) = C$. Define by the relativized s_n^m-theorem a recursive function $\alpha(i)$ as the index of the B-recursively enumerable set defined by

$$W_{\alpha(i)}^B = W_i \cup \bigcup_{x \in f(W_i)} W_x^B.$$

We show $\alpha^{-1}(\mathrm{Ind}_B(E)) = \mathrm{Ind}(FS_C)$, i.e., $\mathrm{Ind}(FS_C) \leqslant_m \mathrm{Ind}_B(E) \equiv_m B''$ and the statement follows immediately.

If $i \in \mathrm{Ind}(FS_C)$, then W_i is finite and $W_i \subseteq C$. Hence $f(W_i)$ is finite and $f(W_i) \subseteq \mathrm{Ind}_B(E)$, i.e., W_x^B is finite for every $x \in f(W_i)$. Since a finite union of finite sets is a finite set, $W_{\alpha(i)}^B$ is finite, i.e., $\alpha(i) \in \mathrm{Ind}_B(E)$.

If $i \notin \mathrm{Ind}(FS_C)$, then (1) W_i is infinite or (2) $W_i \nsubseteq C$. Clearly, if W_i is infinite, then $W_{\alpha(i)}^B$ is infinite, too, and we get $\alpha(i) \notin \mathrm{Ind}_B(E)$. In the second case we get $f(W_i) \nsubseteq \mathrm{Ind}_B(E)$. Hence there exists some $x \in f(W_i)$ such that W_x^B is infinite, i.e., $W_{\alpha(i)}^B$ is infinite, and we get $\alpha(i) \notin \mathrm{Ind}_B(E)$ in this case, too. □

Now we shall show that $B' \equiv_m \text{Ind}(S_C)$ if C is immune. Observe that $S_C = FS_C$ for every immune set C. This fact will be the key to the proof.

Lemma 11. *If C is immune, then $B' \equiv_m \text{Ind}(S_C)$.*

Proof. We proceed by contradiction. Suppose that there exist an immune set C and some $B \subseteq \mathbb{N}$ such that $B' \equiv_m \text{Ind}(S_C)$. We shall show that the assumption yields $\text{Ind}(S_{B'}) \leqslant_m \text{Ind}(FS_{B'})$.

Since $C \leqslant_m \text{Ind}(S_C) \equiv_m B'$, there exist recursive functions f and g with $f^{-1}(B') = C$ and $g^{-1}(\text{Ind}(S_C)) = B'$. With the help of the s_n^m-theorem define a recursive function $\beta(i)$ to be the index of the r.e. set defined by

$$W_{\beta(i)} = \bigcup_{x \in g(W_i)} f(W_x).$$

We show $\beta^{-1}(\text{Ind}(FS_{B'})) = \text{Ind}(S_{B'})$.

If $i \in \text{Ind}(S_{B'})$, then $W_i \subseteq B'$. Hence $g(W_i) \subseteq \text{Ind}(S_C)$, i.e., $\bigcup_{x \in g(W_i)} W_x \subseteq C$. Since the union of the W_x is r.e. and C is immune, the union of the W_x must be a finite set. Thus

$$W_{\beta(i)} = \bigcup_{x \in g(W_i)} f(W_x) = f\Big(\bigcup_{x \in g(W_i)} W_x\Big)$$

is finite.

$\bigcup_{x \in g(W_i)} W_x \subseteq C$ implies $W_{\beta(i)} \subseteq f(C)$. Since $f(C) \subseteq B'$ we get $W_{\beta(i)} \subseteq B'$ and $W_{\beta(i)}$ is finite. Thus $\beta(i) \in \text{Ind}(FS_{B'})$.

If $i \notin \text{Ind}(S_{B'})$, then $W_i \nsubseteq B'$. Hence $g(W_i) \nsubseteq \text{Ind}(S_C)$, i.e., there exists $x \in g(W_i)$ such that $W_x \nsubseteq C$ and therefore $f(W_x) \nsubseteq B'$. Thus $W_{\beta(i)} \nsubseteq B'$, i.e., $\beta(i) \notin \text{Ind}(FS_{B'})$.

In summary $\text{Ind}(S_{B'}) \leqslant_m \text{Ind}(FS_{B'})$. Applying Lemmas 9 and 10 yields $\overline{B''} \equiv_m \text{Ind}(S_{B'}) \leqslant_m \text{Ind}(FS_{B'}) \leqslant_m B''$—a contradiction to the fact that $\overline{B''}$ and B'' are incomparable with respect to many–one reducibility. □

Combining Lemmas 8 and 11 we get

Theorem 12. *If $\text{Ind}(M) \in \Sigma_2^B \backslash \Pi_2^B$, then*
(i) $\text{Ind}(S_C) \in \Sigma_2^B \backslash \Pi_2^B$, *and*
(ii) $\text{Ind}(S_C)$ *is not Σ_2^B-complete, where $C = \text{ind}(M)_{\min}$.*

Proof. (i) C is immune and $C \in \Sigma_2^B \backslash \Pi_2^B$ by Lemma 8. Hence $\text{Ind}(S_C) = \text{Ind}(FS_C) \leqslant_m B'' \in \Sigma_2^B$ by Lemma 10. Furthermore $\text{Ind}(S_C) \notin \Pi_2^B$ since $C \leqslant_m \text{Ind}(S_C)$ and $C \notin \Pi_2^B$.

(ii) Follows immediately by Lemma 11 (otherwise $(B')' \equiv_m \text{Ind}(S_C)$). □

By Theorem 12 we get index sets on every level >1 of the arithmetical hierarchy which are not complete on that level. To see this fix some $n \geq 2$ and define $M_n = FS_{\emptyset^{(n)}}$, where $\emptyset^{(n)}$ is the n-th jump of $\emptyset$ defined as in [9]. Then $\text{Ind}(M_n) \equiv_m \emptyset^{(n)}$ by Lemma 10, i.e., $\text{Ind}(M_n) \in \Sigma_n \setminus \Pi_n$. Define $B = \emptyset^{(n-2)}$. Then $\text{Ind}(M_n) \in \Sigma_2^B \setminus \Pi_2^B$ so that by Theorem 12, $\{i \mid W_i \subseteq \text{Ind}(M_n)_{\min}\}$ is an index set in $\Sigma_2^B \setminus \Pi_2^B = \Sigma_n \setminus \Pi_n$ which is not Σ_n-complete. Accordingly the complement $\{i \mid W_i \nsubseteq \text{Ind}(M_n)_{\min}\}$ is an index set in $\Pi_n \setminus \Sigma_n$ which is not Π_n-complete.

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